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Dualities and Collineations of Projective and Polar Spaces and of Related Geometries

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Preface

The theory of *Buildings*, founded by Jacques Tits, was initially meant to provide a geometric interpretation of exceptional Lie groups, but it was a perfect home for all groups of Lie type, Chevalley groups, twisted or not, isotropic simple algebraic groups, mixed analogues and classical groups. The action of a group on a permutation module is a very strong tool in studying a given group. In this thesis, we want to contribute to the study of groups acting on spherical buildings by investigating the displacements of actions on projective spaces and polar spaces. This will be done by considering two complementary points of view.

In Part I we consider some important classes of finite geometries and obtain relations between their parameters and the various possible displacements of the elements of the geometry for a given automorphism. Our results could be viewed as far-reaching generalizations of a result of Benson, who found such a relation for finite generalized quadrangles. In fact, all our formulae are inspired by Benson's result in that our proofs go along the same lines, but we have to add some additional ideas. One reason is because we deal with geometries of larger girth. Another reason is that, under the notion *automorphism*, one has to understand both type-preserving and non-type-preserving permutations that preserve the structure. The former are the *collineations*, examples of the latter are the *dualities*. A motivating factor for our work is that dualities do not seem to have been thoroughly investigated in full generality in the literature (in contrast to the *polarities*, which are the dualities of order 2, or *trialities*, which are non type-preserving automorphisms of order 3). Benson's original formula only works for collineations.

Which geometries do we consider? We start with the obvious generalizations of generalized quadrangles, namely, the generalized polygons. Then we consider block designs. The reason for this is that collineations in certain buildings (for example projective spaces, generalized quadrangles of order (s, s) or $(q-1, q+1)$ and hexagons of order (s, s), polar spaces of parabolic or symplectic type) are equivalent to dualities in symmetric designs. This leads us to consider a class of near hexagons. The obvious generalization of this class leads us to partial geometries and partial quadrangles. Since we are primarily interested in dualities, we take a closer look at the self-dual examples.

In Part II we go in the opposite direction. Whereas in Part I, we start with a given automorphism and look how the displacements of elements behave, we now start with an assumption on the displacement and try to say something about the automorphisms. The inspiration for our main assumptions stems from a result of Abramenko & Brown, stating that every nontrivial automorphism in a non-spherical building has unbounded displacement, and of Leeb, who proves that in a spherical building, there is always at least one simplex mapped to an opposite one. In spirit, we try to classify those automorphisms which map as few simplices as possible to opposite ones. This is effected by requiring that certain types of simplices are never mapped onto opposites. The most general such assumption is obtained by requiring the above for *chambers*, in which case the automorphism is said to be *domestic*. The ultimate goal is then to classify domestic automorphisms of a given building. We give a completely satisfying classification for the cases of projective spaces and generalized quadrangles. Partial results are obtained for polar spaces and other generalized polygons. Interesting open problems emerge, and also a rather mystic observation—the only known domestic collineations of generalized polygons which map at least one point and at least one line to an opposite point and line, respectively, all have order 4, and we know precisely five of these, up to conjugation—calls for a deeper explanation, which we fail to see at the moment.

Finally, we remark that we restricted ourselves to the classes of buildings of classical type, together with those of rank 2. But our methods should also work for exceptional types and probably give other nice connections and results.

Contents

0

Introduction, definitions and known results

0.1 Incidence structures

In this thesis, we will be concerned with certain classes of incidence structures. In general, an *incidence structure* or *incidence geometry* of *rank* n is an n-partite graph with given classes. Each of these classes is assigned a name or symbol called the *type* of the class, or the *type* of the elements of the class. Two elements that are adjacent in this graph are called *incident*. Depending on the context, one requires some general additional conditions, such as on the valency of elements, or on the girth, or on the diameter, etc. We will be mainly interested in the rank 2 case for Part I, and in finite rank for Part II, where the incidence structures will be heavily related to Tits-buildings.

Let Γ be an incidence structure of rank n, $n \in \mathbb{N}$. When we talk about the *elements* of Γ, then we mean the vertices of the underlying n-partite graph. Sometimes, the elements will also be named after their types, e.g., the *points*, when the type of these elements is "point", or i-*spaces* when the type is i (and then, usually, i also refers to some "dimension"). A *flag* of Γ is a set of pairwise incident elements (here, a flag is allowed to be the empty set, or to be a singleton). A *chamber* is a flag which contains an element of each type. As a condition to be called "geometry", one sometimes requires that in an incidence structure every flag is contained in a chamber. This will always be the case in the examples we consider in this thesis. To emphasize this, we will sometimes use the notion of *chamber geometry* for an incidence structure in which every flag is contained in at least one chamber.

We will now introduce some general notions and then, in the subsequent sections, specialize to particular classes of incidence structures.

Let Γ be an incidence structure of rank n with partition classes $P_1, P_2, \ldots, P_n$. Usually, we view one of the sets P_i, $i \in \{1, 2, \ldots, n\}$, as the point set, and any other element is identified with its point set. This way, we think of an incidence structure as set of points endowed with *subspaces*. The underlying graph of Γ is called the *incidence graph*. A *collineation* of the incidence structure Γ is a graph automorphism of the incidence graph that preserves each class P_i, $i \in \{1, 2, \ldots, n\}$. An *automorphism* of Γ is a graph automorphism that interchanges the classes amongst themselves. So, a collineation is in fact a type-preserving automorphism. A *duality* is an automorphism that induces a non-trivial involution on the n classes. A *polarity* is a duality of order 2. For a given duality σ, an *absolute element* is an element which is incident with its image under σ. Note that this implicitly implies that the type of an absolute element is not fixed under σ.

An incidence geometry Γ is called *thick* if every flag which is not a chamber is contained in at least three chambers. The *rank* of a flag is the number of elements it contains, and the *corank* is the number of elements it falls short to be a chamber (it equals n minus the rank). There is some inductive mechanism that sometimes allows to reduce things to geometries of lower rank. Consider an arbitrary flag F of corank j in Γ, $1 \leq j \leq n$. Then we define the incidence geometry $\mathrm{Res}_\Gamma(F)$, called the *residue of F in* Γ, as the incidence geometry with as elements those elements x of Γ incident with all elements of F, with incidence as in Γ and with natural type set induced by Γ. Clearly, the rank of $\mathrm{Res}_\Gamma(F)$ is equal to the corank of F. It is also clear that, if Γ is thick, then every residue of rank at least 2 is thick.

In a chamber geometry, two chambers are called *adjacent* if they differ in exactly one element. The graph thus constructed will be called the *chamber graph* of the geometry. The distance function in the chamber graph will be denoted by δ. To distinguish it linguistically from other distances, such as distance in the incidence graph, we sometimes refer to it as the *gallery distance*, a word borrowed from the theory of buildings.

If one wants to define "isomorphisms" between incidence geometries, then one needs two incidence geometries over the same type set. An *isomorphism* is then a type preserving bijection. Two incidence structures (necessarily over the same type set) are called *isomorphic* if there exists some isomorphism between them.

When the rank of a geometry is equal to 2, then we often talk about *point-line geometries*. The types are then the "points" and the "lines". A point-line geometry is denoted by a triple $(\mathcal{P}, \mathcal{L}, \mathtt{I})$, where $\mathcal{P}$ is the point set, $\mathcal{L}$ is the line set, and $\mathtt{I}$ is the symmetric incidence relation. In this case, there is a natural *dual* geometry, namely $(\mathcal{L}, \mathcal{P}, \mathtt{I})$, and it is obtained by simply interchanging the names of the types. A point-line geometry which is isomorphic to its dual is called a *self-dual* geometry. Equivalently, a self-dual point-line geometry is a point-line geometry admitting a duality. Similarly, a *self-polar* point-line geometry is one that admits a polarity.

If every line of a point-line geometry Γ is incident with a constant number of points, say $s+1$, and every point is incident with a constant number of lines, say $t+1$, then we say that Γ has *order* or *parameters* (s,t). Here, s and t are cardinal numbers and could be infinite.

A very helpful notion in this thesis is the one of the *double* 2Γ of a point-line geometry Γ. This is defined as follows. The points of 2Γ are the elements of Γ, and the lines of 2Γ are the chambers of Γ. Incidence is induced by the natural inclusion relation. Every line of the geometry 2Γ is incident with exactly two points. If Γ has order (t,t), then 2Γ has order $(1,t)$.

Most point-line geometries we are going to deal with have the property that every line is determined by the set of points incident with it (no "repeated lines"). In that case, we can view the geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ as the set of points $\mathcal{P}$ endowed with a set $\mathcal{B}$ of subsets of $\mathcal{P}$, where $\mathcal{B} = \{B \subseteq \mathcal{P} : (\exists L \in \mathcal{L})(x \in B \Leftrightarrow x \mathtt{I} L)\}$. In this case, there is no need for an incidence relation and we denote $\mathcal{S} = (\mathcal{P}, \mathcal{B})$. Related to this point of view is the *point graph* of $\mathcal{S}$, which is the graph with vertex set $\mathcal{P}$ and adjacency given by *collinearity* (two points x, y are collinear, in symbols $x \sim y$, if they are distinct and incident with a common line) with *multiplicity* the number of lines through x, y. We will denote the distance function in the point graph by d. Dually, there is the *line graph* of $\mathcal{S}$ encoding *concurrency* of lines (two lines are concurrent if they are distinct and incident with a common point). Also related with the point of view of $(\mathcal{P}, \mathcal{B})$ are expressions like "a point lies on a line", "a line goes through a point" (as we already used above), etc. An *adjacency matrix* of $\mathcal{S}$ is an adjacency matrix of the point graph (as a graph with multiple edges). Thus, it is a $v \times v$ matrix (with v the number of points of $\mathcal{S}$) whose rows and columns are indexed by the points of $\mathcal{S}$, and the entry on the place (x, y), $x, y \in \mathcal{P}$, is equal to the number of lines through x, y if $x \neq y$, and to 0 is $x = y$. If there is exactly one line through two points x, y, then we sometimes denote that line by xy. For a point x, the set of points collinear with x, completed with x itself, will be denoted by $x^\perp$.

When we talk about the *distance* of two elements in a point-line geometry, we will always mean the graph-theoretical distance measured in the corresponding incidence graph. Hence, if we consider distance in the point graph, then we will always explicitly mention this.

The prototype of an incidence geometry of rank n is a projective space of dimension n over a skew field $\mathbb{K}$. Here, the elements are the nontrivial proper subspaces of the projective space (or, equivalently, those of the underlying vector space), and incidence is inclusion made symmetric. We will always define the type of an element as its projective dimension. Hence the type set equals $\{0, 1, 2, \ldots, n-1\}$. We will also use the standard notation $\mathsf{PG}(n, \mathbb{K})$ for this projective space.

We also use standard notation with respect to subspaces. For instance, the subspace generated by a set S is denoted by $\langle S \rangle$.

Let θ be a duality of a projective space $\mathsf{PG}(n, \mathbb{K})$, with $\mathbb{K}$ a skew field. Recall that an absolute element U is a subspace which is incident with its image U^θ. A *symplectic polarity*, or *null-polarity*, is a polarity for which every point is absolute. Then necessarily $\mathbb{K}$ is a commutative field, n is odd, and θ is related to a non-degenerate alternating bilinear form.

0.2 Generalized polygons

0.2.1 Definitions

For the following definitions and claims we refer the reader to Chapter 1 of [55] and to [38].

One of the most important classes of rank 2 geometries is constituted by the generalized polygons. A *generalized n-gon* $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a point-line geometry such that the diameter of the incidence graph is n and its girth is $2n$. Here, $n \geq 2$, and for $n = 2$, the incidence graph is a complete bipartite graph and hence $\mathcal{S}$ is a trivial geometry in which every point is incident with every line. Nevertheless these geometries are prominently present in incidence geometry. For example, every rank 2 residue in a projective space is either a projective plane (i.e., a projective space of dimension 2 or, equivalently, a thick generalized 3-gon) or a generalized digon. But as a research object in incidence geometry, generalized digons are too simple. So we will usually assume $n \geq 3$.

A generalized n-gon of order $(1, 1)$ is called an *ordinary n-gon*. Often we replace n-gon by a linguistic expression such as *triangle* for 3-gon, *quadrangle* for 4-gon, *pentagon* for 5-gon, *hexagon* for 6-gon, *heptagon* for 7-gon, *octagon* for 8-gon, *decagon* for 10-gon and *dodecagon* for 12-gon. We also often forget the adjective “generalized”, if no confusion can arise. As already mentioned above, thick generalized triangles are projective planes and vice versa.

A thick generalized polygon always has an order, and a polygon with an order is either thick or the double of a thick one of order (s, s), or the dual of such a double, or an ordinary n-gon. In general, the class of generalized polygons is closed under taking the double. The double of a generalized n-gon is a generalized $2n$-gon. Conversely, every generalized $2n$ gon of order $(1, t)$ is the double of a generalized n-gon of order (t, t), determined up to duality (see next paragraph). More generally, if every line of a generalized polygon has exactly 2 points, then it is a generalized $2n$-gon and the double of a generalized n-gon, possibly without order. Besides doubling, there is also the notion of "multiplying" a geometry, and in particular a generalized n-gon, by some natural number k (and in particular we obtain a generalized kn-gon). Since we will not need this notion, we will not give a precise definition. We just mention that the Structure Theorem of generalized polygons says that every generalized polygon is either thick, or contains 0 or 2 thick elements, or arises as nontrivial multiple of a thick generalized polygon, see [53] and [55], Theorem 1.6.2.

If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a generalized polygon of order (s, t), then we denote the dual geometry by $\mathcal{S}^{\mathrm{dual}} = (\mathcal{L}, \mathcal{P}, \mathtt{I})$. It is also a generalized polygon, now of order (t, s). This gives rise to the *principle of duality*, which states that every theorem in the theory of generalized polygons can be dualized by interchanging the roles of $\mathcal{P}$ and $\mathcal{L}$, and of s and t. If $s = t$, then we say that $\mathcal{S}$ has order s.

Generalized triangles of order (s, t) only exist for $s = t$ and for $s \geq 2$ they are exactly the projective planes of order $s = t$. There is a vast literature on these structures. Finite generalized quadrangles have been investigated in detail; see [38]. We refer to [50] and [55] for extensive surveys on (finite and infinite) generalized polygons. Generalized polygons were introduced by Jacques Tits [51] in 1959. They are the spherical buildings of rank 2. As such they provide the natural geometries for Chevalley groups of rank 2, and, more generally, for algebraic, classical and mixed groups, and twisted versions, all of relative rank 2.

A major result on finite generalized polygons is due to Feit & Higman [21] and states that for a finite generalized n-gon, $n \geq 3$, of order $(s, t) \neq (1, 1)$, we always have either $n = 3$ (and then $s = t$), or $n = 4$ (and, if $s > 1$ and $t > 1$, then $s + t$ divides $st(1 + st)$), or $n = 6$ (and, if $s > 1$ and $t > 1$, then st is a perfect square), or $n = 8$ (and, if $s > 1$ and $t > 1$, then $2st$ is a perfect square), or $n = 12$ (and then $1 \in \{s, t\}$).

Since generalized polygons are the spherical buildings of rank 2, there is a lot of building terminology that we shall use, too. For instance, two elements of a generalized n-gon are called *opposite* if they are at maximal distance in the incidence graph; in this case the distance is n. If two elements x, y are not opposite, then the definition immediately implies that there is a unique element z incident with x and which is closer to y than x is (in the incidence graph). We denote z by $\mathrm{proj}_x y$ and call it the *projection of y onto x*.

A *sub-n-gon* $\mathcal{S}'$ of a generalized n-gon $\mathcal{S}$ is a generalized n-gon the elements of which are also elements of $\mathcal{S}$ and the incidence relation is inherited from $\mathcal{S}$. A sub-n-gon of order 1 is called an *apartment* of $\mathcal{S}$.

Another, rather difficult issue is the existence and uniqueness question for generalized polygons of given order. For small orders, the following is known (for n-gons with $n \geq 4$; we only list relevant cases for us).

(GQ) Generalized quadrangles of order (q, q), (q, q^2), (q^2, q^3) and $(q-1, q+1)$ exist for every prime power q. Conversely, every known thick finite generalized quadrangle admits one of these orders (but of course excluding $(q-1, q+1)$ for $q = 2$), up to duality.

(GQ2) Generalized quadrangles of order $(2, t)$, with t any cardinal number, exist only for $t \in \{1, 2, 4\}$ and are uniquely determined by these parameters (see [38]).

(GQ3) Generalized quadrangles of order $(3, t)$, with t any cardinal number, exist only for $t \in \{1, 3, 5, 9\}$ and are uniquely determined by these parameters, except for $t = 3$, in which case there exist exactly two isomorphism classes dual to each other (see [38]).

(GH) Generalized hexagons of order (q, q) and (q, q^3) exist for every prime power q. Conversely, every known thick finite generalized hexagon admits such an order.

(GH2) Generalized hexagons of order $(2, t)$, with t any finite number, exist only for $t \in \{1, 2, 8\}$ and are uniquely determined by their parameters, except for $t = 2$, in which case there exist exactly two isomorphism classes, dual to each other (see [14]).

(GO) Generalized octagons of order $(2^e, 2^{2e})$, with e an odd natural number, exist. Conversely, all known thick generalized octagons admit these parameters, up to duality.

Now we introduce some additional terminology concerning special kinds of collineations of generalized polygons. If a collineation θ in a generalized $2n$-gon fixes all elements at distance at most n from a given point x (or, dually, line L), then we call θ a *central* *collineation* (or, dually, an *axial collineation*) with *center* x (*axis* L). For $n = 4$, central collineations with center x are also called *symmetries about the point* x. Dually, we have *symmetries about a line*.

Now we highlight some special substructures of generalized polygons. Let $\mathcal{S}$ be a generalized n-gon, $n \geq 3$, and let $\mathcal{S}'$ be a sub-n-gon. Then we call $\mathcal{S}'$ *full* if every point of $\mathcal{S}$ incident in $\mathcal{S}$ with a line of $\mathcal{S}'$ belongs to $\mathcal{S}'$. Dually, one defines an *ideal* subpolygon. A *large* subpolygon is a sub-n-gon with the property that every element of $\mathcal{S}$ is at distance at most $n/2$ from some element of $\mathcal{S}'$. A distance-j ovoid in a generalized $2n$-gon, with $1 \leq j \leq n$, is a set $\mathcal{O}$ of points at mutually distance $\geq 2j$ and such that every element of $\mathcal{S}$ is at distance $\leq j$ from at least one element of $\mathcal{O}$. Dually, we have the notion of a distance-j spread. For $j = n$, a distance-n-ovoid is usually simply called an *ovoid*, and similarly for *spread*. It is well known that the set of absolute points of any polarity in any generalized $2n$-gon is an ovoid, and, dually, the set of absolute lines is a spread.

Now let $\mathcal{S}$ be a generalized quadrangle. An *ovoidal subspace* $\mathcal{O}$ of $\mathcal{S}$ is a set of points such that every line either shares all its points with $\mathcal{O}$, or is incident with exactly one point of $\mathcal{O}$. Another name for this structure is a *geometric hyperplane*. If $\mathcal{S}$ is a generalized hexagon, then an *ovoidal subspace* $\mathcal{O}$ of $\mathcal{S}$ is a set of points such that every other point is at distance 2 from a unique point of $\mathcal{O}$. Ovoidal subspaces of generalized hexagons were introduced and classified in [7], see also [23, 24]. They are the ovoids, the point sets obtained from a line L by considering all points at distance at most 3 from L, and the large full subhexagons. In Section 7.3 we will generalize the definition of ovoidal subspace to all generalized $2n$-gons and prove a similar classification.

0.2.2 Examples

The main examples of generalized polygons arise from algebraic groups of relative rank 2 in the broad sense. We now mention some specific examples and classes that we will encounter in the course of this thesis.

For each field $\mathbb{K}$ there is the *symplectic generalized quadrangle* $\mathsf{W}(\mathbb{K})$ arising as the set of absolute points and fixed lines of a symplectic polarity in $\mathsf{PG}(3, \mathbb{K})$. For finite $\mathbb{K}$ of order q, it is denoted by $\mathsf{W}(q)$ and it has order q. The quadrangle $\mathsf{W}(2)$ is the smallest thick generalized quadrangle. It is self-dual and even self-polar. In fact, $\mathsf{W}(\mathbb{K})$ is self dual if and only if $\mathbb{K}$ is a perfect field of characteristic 2, and it is self-polar if and only if $\mathbb{K}$ admits a so-called *Tits automorphism*, which is a square root of the Frobenius automorphism (in general, the *Frobenius automorphism* sends x to x^p, with p the characteristic of the field).

In Chapter 6 we will have to deal with the unique generalized quadrangle of order $(3,5)$. In general, a quadrangle of order $(q-1, q+1)$ for q even can be constructed as follows. Consider $\mathsf{PG}(3,q)$ and a plane π in which we consider a *hyperoval* (i.e., a set of $q+2$ points no three of which collinear). The points of the quadrangle are the points off π, and the lines are the lines of $\mathsf{PG}(3,q)$ meeting the hyperoval in exactly one point.

Concerning generalized hexagons, the *split Cayley hexagons* play a similar role in the theory of hexagons as the symplectic quadrangles in the theory of quadrangles. They are also defined for every field $\mathbb{K}$ and denoted by $\mathsf{H}(\mathbb{K})$. They are constructed on the parabolic quadric $\mathsf{Q}(6,\mathbb{K})$, but we will not need an explicit construction. We content ourselves with mentioning that they are self dual if and only if $\mathbb{K}$ is a perfect field of characteristic 3, and they are self-polar if and only if $\mathbb{K}$ admits a Tits automorphism. Hence the smallest split Cayley generalized hexagon $\mathsf{H}(2)$ is not self-dual and so there are two smallest generalized hexagons. But $\mathsf{H}(3)$ is self-dual and even self-polar.

All known thick finite generalized hexagons are related to *triality*, which is a duality of order 3 on the geometry of the hyperbolic quadric $\mathsf{Q}^+(7,q)$.

Also, all known finite generalized octagons are related to Ree groups in characteristic 2.

The starting point of Part I of this thesis is Benson's formula for finite generalized quadrangles, which states a connection between the number of fixed points and the number of points mapped to a collinear point with respect to a collineation of a generalized quadrangle. If that quadrangle has order (s,t), and if that collineation has f_0 fixed points and f_1 points mapped onto a collinear point (and note that by definition this does not include fixed points), then Benson proved in [5] that $(1+t)f_0 + f_1 \equiv 1 + st \mod (s+t)$. It is precisely this formula that we shall try to generalize, both to other generalized polygons, and to dualities.

0.3 Polar spaces

0.3.1 Definitions

A *polar space* Γ *of rank* n, $n \geq 2$, is a chamber geometry of rank n with type set $\{0, 1, \ldots, n-1\}$ satisfying the following axioms (see Chapter 7 of [52] or [58]), where we call the elements of type 0 *points*.

(PS1) The elements of type $< i$ incident with any element of type i form naturally a projective space of dimension i in which the type of an element in Γ is precisely the dimension of the corresponding subspace in that projective space.

(PS2) Every element of Γ is determined by the set of points incident with it and the point sets of two elements of Γ intersect in a subspace of each.

(PS3) For every point x and every element M of type $n-1$ not incident with x, there exists a unique element M' through x of type $n-1$ whose point set intersects the point set of M in the point set of an element of type $n-2$. Also, no element of type 1 is incident with x and a point of M unless it is incident with M' or coincides with M'.

(PS4) There exist two elements of type $n-1$ not incident with any common point.

Axiom (PS1) justifies the following terminology: we call elements of type i *i-dimensional subspaces*. Also, 1-dimensional subspaces are simply called *lines*, 2-dimensional ones *planes* and $n-1$-dimensional ones *maximal subspaces*. The *codimension* of an i-dimensional subspace is by definition equal to $n-2-i$. Two points that are incident with a unique common line will be called *collinear*, and we will thus also use the notation $x^{\perp}$ for the set of points collinear to the point x completed with x itself. In such a way we can consider a polar space as a point-line geometry by "forgetting" the i-dimensional subspaces for $i \geq 2$. These can always be reconstructed merely using the points and lines. In this setting, it is natural to see the subspaces as sets of points, and we will indeed take this point of view. This way we can talk about the intersection of subspaces.

If x is a point and L is a line of a polar space Γ, x not on L, then considering a maximal subspace M incident with L (we also say *through* L), and applying Axiom (PS3), we see that

(BS) either all points on L are collinear with x, or exactly one point on L is collinear with x.

A major result of Beukenhout & Shult [9] is that this observation—known as the *Buekenhout-Shult one-or-all axiom*—along with some nondegeneracy conditions such as (1) every line contains at least three points, (2) no point is collinear with all other points, and under a suitable condition that bounds the rank, characterizes the class of polar spaces. The simplicity of Axiom (BS) played a major role in the success of studying polar spaces and, in fact, we will also use that axiom as a central property of polar spaces. Moreover, the above motivates the notion of a *degenerate polar space* as a point-line geometry in which Axiom (BS) holds, but which is not the restriction to points and lines of a polar space. With "polar space" we will never include the degenerate ones, except when explicitly mentioned (for example, we sometimes say "possibly degenerate polar space").

Polar spaces of rank 2 are just thick generalized quadrangles or generalized quadrangles with 2 lines through every point (the so-called *grids*). As soon as the rank is at least 3, then there is a classification due to Tits [52]. Roughly, this classification says that a polar space of rank at least 4 arises from a bilinear, sesquilinear or pseudo-quadratic form in some vector space. In the rank 3 case there is one other class of polar spaces parametrized by octonion division rings (here, the planes of the polar space are projective planes over alternative division rings).

Note that the chambers of a polar space look very much like the chambers of a projective space: they are sets of nested projective subspaces of dimension 0 up to $n-1$. But gallery distances reach higher values, since there are disjoint maximal subspaces.

The notion of "geometric hyperplane", that we mentioned in the previous section for generalized quadrangles, can be generalized for polar spaces. First we can define a *geometric subspace* of a polar space as a set of points such that, if two collinear points x and y belong to that set, then all points of the line xy belong to it. We often view geometric hyperplanes as substructures endowed with all subspaces completely contained in it. In order to explicitly distinguish between geometric subspaces and ordinary subspaces, we sometimes call the latter *projective subspaces*. Now, a *geometric hyperplane* is a geometric subspace with the property that every line contains at least one point of it, and at least one line contains exactly one point of it. It is easy to show that geometric subspaces are (possibly degenerate) polar spaces. The *corank* of a geometric subspace equals i if every i-dimensional (projective) subspace meets it in at least one point, and there exists an i-dimensional subspace meeting it in exactly one point. Hence, geometric hyperpanes are the geometric subspaces of corank 1.

If we understand with *distance between two elements* the graph-theoretical distance between them in the incidence graph, then this notion does not fully cover all the possible *mutual positions* of two elements (by which we mean the isomorphism classes of the substructures induced by all shortest paths between them in the incidence graph). For instance, for two lines, there are six possible mutual positions given by (1) equality, (2) being contained in a common plane, (3) intersecting in a unique point but not contained in a plane, (4) being disjoint but some plane contains one of them and intersects the other in a point, (5) being disjoint and no plane containing one of them intersects the other in a point, (6) both contained in a common projective subspace, but not in a plane. Clearly, in the cases (2), (3) and (6) the lines are at distance two from each other. But for points, it does. Two points can have only three possible mutual positions, given by the distances $0, 2, 4$ in the incidence graph. A special mutual position is *opposition* given by the maximum distance between two elements of the same type. It is characterized as follows: two subspaces U and U' of dimension i are opposite if and only if no point of U

is collinear with all points of U'. It follows that this relation is symmetric.

We now generalize the notion of "projection" in polygons to polar spaces. Our definition is in conformity with the definition of projection in buildings (see next section), where the projection of a flag F onto another flag F' is the intersection of all chambers appearing as last chamber in a minimal gallery connecting F with F' (i.e., the first chamber contains F and the last one F'). Since F' is, however, always in that final chamber, one usually does not mention it. Let U and V be two projective subspaces of a polar space Γ, and suppose that they are neither opposite nor incident (otherwise the projection is empty). Then $\mathrm{proj}_U V$ is the set (a flag) of the following subspaces: the intersection $V \cap U$, if not empty; the set of points of U collinear with all points of V, if not empty and if it does not coincide with U; the unique minimal subspace containing all points of U and all points of V that are collinear with all points of U, if it does not coincide with U. At least one of these subspaces is well defined. In the generic case, all these subspaces are distinct and hence $\mathrm{proj}_U V$ is a set of three subspaces: two contained in U and one containing U. We will make the following agreement: with $\mathrm{proj}_{\subseteq U} V$ we mean the set of points of U collinear with all points of V (and this time it could coincide with U, with $V \cap U$ or with the empty set) and with $\mathrm{proj}_{\supseteq U} V$ we shall denote the subspace generated by U and the set of points of V that are collinear with all points of U (and this time, this could also coincide with U).

0.3.2 Examples

As already mentioned, polar spaces of rank at least 3 arise from certain forms in vector spaces, except for one class of so-called "non-embeddable polar spaces". There is one particular class of polar spaces that we will encounter and these are the *symplectic polar spaces*. They arise as the sets of absolute and fixed subspaces of dimension at most n of symplectic polarities in $\mathsf{PG}(2n+1, \mathbb{K})$, with $\mathbb{K}$ any field, and they are denoted by $\mathsf{W}(2n+1, \mathbb{K})$.

Another special class of examples is the class of the hyperbolic quadrics. These are polar spaces denoted $\mathsf{Q}^+(2n+1, \mathbb{K})$ of rank $n+1$ arising from the subspaces contained in a quadric in $\mathsf{PG}(2n+1, \mathbb{K})$ with equation $X_0X_{2n+1} + X_1X_{2n} + \cdots + X_nX_{n+1} = 0$. They have a rather special well known property, namely, that the maximal subspaces come in two classes, and members of the same class meet in a projective subspace of even codimension whereas members of distinct classes meet in subspaces of odd codimension. It follows that every $(n-1)$-dimensional subspace is contained in exactly 2 maximal subspaces. This implies that the geometry is not thick.

With $\mathsf{Q}^+(2n+1,\mathbb{K})$, one can associate another geometry, called its *oriflamme geometry*, denoted by $\mathsf{D}_{n+1}(\mathbb{K})$ and defined as follows. The type set of this thick chamber geometry of rank $n+1$ is $\{0,1,\ldots,n-2,n_+,n_-\}$. The elements are the subspaces of $\mathsf{Q}^+(2n+1,\mathbb{K})$ of dimension distinct from $n-1$, and their type is their dimension if this is not equal to n. If the dimension of the subspace is equal to n, then we assign the type n_+ to one class of maximal subspaces and n_- to the other. This is indeed a thick chamber geometry of rank n, see [52].

0.4 Buildings and domestic automorphisms

In this thesis, we will not work with general buildings, but we will be concerned with special classes. We will view these classes as classes of incidence geometries. They are the ones that we have introduced so far. However, the main problems of this thesis, and in particular those of Part II, can be asked in general for arbitrary spherical buildings.

Hence we shall not need the general definition of a building, not even of a spherical building. But we do borrow some terminology. This terminology has been introduced above and comprises notions like *chamber, apartment, gallery, gallery distance*.

The central notion of Part II of this thesis is that of a *J-domestic automorphism*. And this can only be fully understood if one considers this in full generality for arbitrary spherical buildings. This is what we are going to do now, although we did not define the notion of a building. We refer to [1, 39, 53, 60] for more details on buildings.

One special feature about spherical buildings that we will need is that of opposition. Opposition has to do with maximal distance, and for the distance between flags, one has several ways to define this. For instance flags are cliques in the incidence graph, and one could look at the graph-theoretic distance between these cliques. Or one could view flags as sets of all chambers it is contained in, and then look at the distance between these sets of chambers in the chamber graph. The latter point of view is most common for buildings, where the notion of a chamber is more central than that of an element. Consequently, two flags are opposite if one is at maximal distance from the other, with the distance defined by the chamber graph. If we restrict to vertices, then this opposition relation defines a permutation of the type set I. The types of arbitrary opposite flags are then in accordance with this permutation. Therefor, we call this permutation also *opposition* and corresponding type sets are called *opposite*. A set J of types is called *self-opposite* if it is opposite itself.

Let Ω be a spherical building, and let θ be an automorphism of Ω. We emphasize that

θ does not need to be type-preserving. Then we call θ *domestic* if no chamber of Ω is mapped onto an opposite chamber. More in particular, for a subset T of the type set of Ω, we say that θ is *T-domestic*, if θ does not map any flag of type T onto an opposite one.

Of course, one sees that, if the image under θ of T is not the same as the image under opposition of T (for instance if θ is type-preserving and T is not self-opposite), then θ is automatically T-domestic. One could resolve this by replacing, in the definition of T-domestic, "being opposite" by "being of the same type T and at maximal distance", but we choose not to do so. Instead, we only consider type sets T which have the same image under opposition as under θ. However, both solutions are basically equivalent as will be illustrated in Chapter 5 for the case of projective spaces.

The opposition relation on types acts trivially for polar spaces, generalized $2n$-gons and the oriflamme complexes of hyperbolic quadrics of even rank, and it is nontrivial for projective spaces, generalized $(2n+1)$-gons and the oriflamme complexes of hyperbolic quadrics of odd rank.

If the elements of type $i \in I$ of a building Ω have a short special name, such as "point" or "line", then we sometimes refer to $\{i\}$-domestic as *point-domestic* or *line-domestic*, respectively. Similarly for short combinations, such as {point-line}-domestic. We will also usually write i-domestic instead of $\{i\}$-domestic. In such terminology domestic would be equivalent with chamber-domestic.

It is a general feature of the geometries related to buildings that all residues are also buildings. In particular, the residues in polar spaces of elements are direct sums of projective spaces and polar spaces. We deduce that, for two opposite subspaces U and V of dimension $i \leq n-3$ in a polar space Γ of rank n, the geometry consisting of all subspaces of Γ all of whose points are collinear with all points of U and V, with the type their dimension, is a polar space of rank $n-i-1$. This observation will sometimes be used allowing for an inductive setting.

0.5 Near polygons

A *near polygon* is a partial linear space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ with the following property: if x is a point and L is a line not incident with x, then there exists a unique point y incident with L for which $\mathrm{d}(x, y)$ is minimal. If the maximal distance between two elements is n, then the near polygon is also called a *near n-gon*. Note that n is necessarily even, since we take distances in the incidence graph.

A detailed account on near polygons is contained in [16], to which we refer for constructions, characterizations and classifications. The definition is due to Shult & Yanushka [42].

The standard examples of near $2n$-gons arise from polar spaces of rank n by taking as point set the set of maximal subspaces and collinearity is intersecting in an $(n-2)$-dimensional subspace. However, the near polygons that we will encounter and consider are the point-line incidence structures that arise from taking the double of other geometries, in this case symmetric designs, symmetric partial geometries and symmetric partial quadrangles. In the next sections we give some details about this.

0.6 Designs

0.6.1 Definition

A $2-(v, t+1, \lambda+1)$-*design*, $v, t, \lambda \in \mathbb{N}$, with $v > t+1$ and $t \geq 1$, is an incidence structure $\mathcal{D} = (\mathcal{P}, \mathcal{B}, \mathtt{I})$, with $\mathcal{P}$ a set of points, $\mathcal{B}$ a set of blocks and $\mathtt{I}$ the incidence relation, which satisfies the following conditions:

1. $|\mathcal{P}| = v$;
2. Any two distinct points are incident with exactly $\lambda + 1$ blocks;
3. $|B| = t + 1$ for any block B.

We will call such a design *symmetric* if $|\mathcal{P}| = |\mathcal{B}|$. In this case, any two distinct blocks meet in exactly $\lambda+1$ points and every point is contained in exactly $t+1$ blocks. Hence the above three conditions are also satisfied for the dual incidence structure. This explains the name “symmetric”, see [30].

In the case of a $2-(v, t+1, \lambda+1)$-design, the point graph is a (complete) graph with multiple edges (two distinct vertices are always joined by $\lambda+1$ edges). Hence an adjacency matrix A has zero on the diagonal and $\lambda + 1$ elsewhere. It follows easily that A has rank 2 (“rank” in the sense of rank of a matrix).

Examples of 2-designs exist in great numbers. For instance, finite projective spaces give rise to many such designs taking the point set of the design to be to point set of the projective space, and the block set the set of subspaces of certain fixed dimension. Symmetric designs are rarer. We will consider some examples at the end of Chapter 2.

0.6.2 The double of a symmetric design

We observe that the double of a symmetric $2-(v,t+1,\lambda+1)$-design is a near hexagon of order $(1,t)$ for which the following property holds: for every two points x and y which lie at distance 4 from each other, there exist precisely $\lambda+1$ paths of length 4 from x to y. We will say that such a near hexagon is of order $(1,t;\lambda+1)$

0.7 Partial geometries

0.7.1 Definition

Partial geometries were introduced by Bose [6] in 1963 as a geometric approach to many strongly regular graphs. Although a number of classes and sporadic examples of (finite) partial geometries are known, they do not seem to exist in great numbers.

A *(finite) partial geometry* is an incidence structure $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathtt{I})$, with an incidence relation satisfying the following axioms

1. each point is incident with $t+1$ lines $(t\geq 1)$ and two distinct points are incident with at most one line;

2. each line is incident with $s+1$ points $(s\geq 1)$ and two distinct lines are incident with at most one point;

3. if x is a point and L is a line not incident with x, then there are exactly α $(\alpha\geq 1)$ points $x_1,x_2,\ldots,x_\alpha$ and α lines $L_1,L_2,\ldots,L_\alpha$ such that $x\mathtt{I}L_i\mathtt{I}x_i\mathtt{I}L$, $i\in\{1,2,\ldots,\alpha\}$.

We will say that such a partial geometry is of *order* (s,t,α), with $1\leq\alpha\leq\min\{s,t\}+1$. If $|\mathcal{P}|=v$ and $|\mathcal{L}|=b$, then $v=\frac{(s+1)(st+\alpha)}{\alpha}$ and $b=\frac{(t+1)(st+\alpha)}{\alpha}$. If $1<\alpha<\min\{s,t\}$, then we say that $\mathcal{S}$ is *proper*. If $\alpha=1$, then $\mathcal{S}$ is a generalized quadrangle of order (s,t).

In a finite projective plane of order q, any non-void set of l points may be described as an $\{l;n\}$-arc, where $n\neq 0$ is the largest number of collinear points in the set. For given q and n ,$n\neq 0$, l can never exceed $(n-1)(q+1)+1$, and an arc with that number of points will be called a *maximal arc* (cfr. [4]). It is easily seen that a maximal arc meets every line in either 0 or n points.

0.7.2 Partial geometries which arise from maximal arcs

We are able to construct a partial geometry from a maximal arc (cfr. [47]). Suppose that we have a maximal $\{qn - q + n; n\}$-arc K $(1 < n < q)$ of a projective plane π of order q. Define the points of the partial geometry $\mathcal{S}$ as the points of π which are not contained in K. The lines of $\mathcal{S}$ are the lines of π which are incident with n points of K and the incidence is the incidence of π. This gives us a partial geometry of order $(q - n, q - q/n, q - q/n - n + 1)$, which we shall denote by $\mathsf{pg}(K)$.

Consider an ovoid $\mathcal{O}$ and a 1-spread $\mathcal{R}$ of $\mathsf{PG}(3, 2^m)$, $m > 0$, such that each line of $\mathcal{R}$ has one and only one point in common with $\mathcal{O}$. Let $\mathsf{PG}(3, 2^m)$ be embedded as a hyperplane H in $\mathsf{PG}(4, 2^m) = P$, and let x be a point of $P \backslash H$. Call C the set of the points of $P \backslash H$ which are on a line xy, with $y \in \mathcal{O}$. Then the point set C is a maximal $\{2^{3m} - 2^{2m} + 2^m; 2^m\}$-arc of the projective plane π defined by the 1-spread $\mathcal{R}$ (cfr. [47]). We will call such maximal arcs *Thas* 1974 maximal arcs. As described above, we can construct a (symmetric, meaning $s = t$) partial geometry $\mathsf{pg}(C)$ from this arc C having order $(2^{2m} - 2^m, 2^{2m} - 2^m, 2^{2m} - 2^{m+1} + 1)$.

An interesting example of this situation occurs when the spread is a regular spread (so there arises a Desarguesian projective plane of order 2^{2m}) and the ovoid is a Suzuki-Tits ovoid (hence the maximal arc is not a Denniston maximal arc; see [47]).

0.7.3 The sporadic partial geometry of Van Lint & Schrijver

The following partial geometry is due to Van Lint & Schrijver [54]. We base our construction on the construction by Cameron & Van Lint [13].

Let $\mathbb{F}_3$ be the field of order 3 and F_3^6 a coordinate 6-dimensional vector space over $\mathbb{F}_3$. Let W be the 1-dimensional subspace spanned by the all-one-vector. Then the points of the partial geometry are the vectors of the quotient space $\mathbb{F}_3^6/W$ whose representatives have coordinates adding up to $1 \in \mathbb{F}_3$, and the lines are the vectors of the quotient space $\mathbb{F}_3^6/W$ whose representatives have coordinates adding up to $-1 \in \mathbb{F}_3$. A coset $W + v$ is incident with a coset $W + v'$ if and only if $v - v'$ has five identical coordinates.

This partial geometry has an obvious polarity, which we shall refer to as the *standard polarity*, and it is induced by the linear map sending a vector to its negative.

0.7.4 The double of a partial geometry

Let $\mathcal{S}$ be a partial geometry of order (t, t, α). Then the double $2\mathcal{S}$ is a near octagon, that is, a near 8-gon, of order $(1, t)$, for which the following property holds: for every two points x and y which lie at distance 6 from each other, there exist precisely α paths of length 6 from x to y, and for every two points x' and y' which lie at distance 4 from each other there exists precisely 1 shortest path from x' to y'. We will say that such a near octagon is of *order* $(1, t; \alpha, 1)$. Conversely, each near octagon of order $(1, t; \alpha, 1)$ arises from a partial geometry of order (t, t, α).

0.8 Partial quadrangles

0.8.1 Definition and examples

The last class of rank 2 incidence geometries that we consider is the class of partial quadrangles, introduced by Cameron [10]. The literature about partial quadrangles is not extensive, and there are not so many examples. The point graph of a partial quadrangle is a strongly regular graph, just as is the case for partial geometries.

A point-line geometry $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a *partial quadrangle* of order (s, t, μ), with $s \geq 1$ and $t \geq 1$, if

(PQ1) each line contains $s + 1$ points and each point is incident with $t + 1$ lines;

(PQ2) there are no digons and no triangles in $\mathcal{S}$;

(PQ3) two noncollinear points are collinear with precisely μ common points.

The total number of points of a partial quadrangle with order (s, t, μ) is $v = 1 + \frac{(t+1)s(\mu+st)}{\mu}$.

It is well known that the eigenvalues of the point graph are integers, except for the unique partial quadrangle of order $(1, 1, 1)$, the ordinary pentagon. Also if $\mu = t + 1$ then we have a generalized quadrangle. We will now give some examples of partial quadrangles which are not generalized quadrangles.

The examples of partial quadrangles with $s = 1$ are the strongly regular graphs without triangles. These comprise the pentagon, the Petersen graph, the Clebsch graph, the Hoffman-Singleton graph, the Gewirtz graph and the two Higman-Sims graphs on 77 and 100 vertices. We do not need a precise definition of these graphs, as they will only be needed for illustration. We mention that the order of these partial geometries is respectively $(1,1,1)$, $(1,2,1)$, $(1,4,2)$, $(1,6,1)$, $(1,9,2)$, $(1,15,3)$ and $(1,21,6)$.

An infinite class class of examples is provided by generalized quadrangles of order (s,s^2). Consider an arbitrary point x in such a generalized quadrangle $\mathcal{S}$. Then the points of the partial quadrangle $\mathcal{S}'$ are the points of $\mathcal{S}$ opposite x, whereas the lines are the lines of $\mathcal{S}$ at distance 3 from x. Incidence is natural. This gives rise to a partial quadrangle of order $(s-1,s^2,s(s-1))$.

Another class of examples is formed by the following construction. Let $\mathcal{K}$ be a $(t+1)$-cap in $\mathsf{PG}(d,q)$, i.e., a set of $t+1$ points no three of which are collinear, with the property that every point not in $\mathcal{K}$ is incident with exactly $t+1-\mu$ tangents. Embed $\mathsf{PG}(d,q)$ as a hyperplane in $\mathsf{PG}(d+1,q)$ and define the points of the partial quadrangle $T_d^*(\mathcal{K})$ as the points of $\mathsf{PG}(d+1,q) \setminus \mathsf{PG}(d,q)$. The lines are the lines of $\mathsf{PG}(d+1,q)$ not belonging to $\mathsf{PG}(d,q)$ and intersecting $\mathcal{K}$ in a point. Incidence is natural. This gives rise to a partial quadrangle of order $(q-1,t,\mu)$.

A special case occurs when $\mathcal{K}$ is an ovoid of $\mathsf{PG}(3,q)$. In this case, the partial quadrangle is isomorphic to the one derived from the generalized quadrangle $T_3(\mathcal{K})$ (see [38]) using the construction above.

All other such examples are "classified" and belong to the following list: a unique 11-cap in $\mathsf{PG}(4,3)$, a unique 56-cap in $\mathsf{PG}(5,3)$, a possibly unique 78-cap in $\mathsf{PG}(5,4)$ and a possibly existing 430-cap in $\mathsf{PG}(6,4)$. The orders of the corresponding (possibly potential) partial quadrangles are respectively $(2,10,2)$, $(2,55,20)$, $(3,77,14)$ and $(3,429,110)$.

Finally, there are examples arising from hemisystems in generalized quadrangles of order (q,q^2). A *hemisystem* in such a quadrangle is a set of points such that each line contains exactly $\frac{q+1}{2}$ points of that set. Then the partial quadrangle is the "restriction" of the generalized quadrangle to that hemisystem (with obvious meaning). It has order $(\frac{q-1}{2},q^2,\frac{(q-1)^2}{2})$. This was proved by Thas for the generalized quadrangles of order (q,q^2) arising from Hermitian varieties see [49] and by Cameron, Delsarte and Goethals for generalized quadrangles of order (q,q^2) [11].

0.8.2 The double of a partial quadrangle

Let $\mathcal{S}$ be a symmetric partial quadrangle, i.e., a partial quadrangle of order (t,t,μ). Then the double $2\mathcal{S}$ is a near decagon, that is, a near 10-gon, of order $(1,t)$, for which the following property holds: for every two points x and y which lie at distance 8 from each other, there exist precisely μ paths of length 8 from x to y, and for every two points x' and y' which lie at distance 4 or 6 from each other there exists precisely 1 shortest path from x' to y'. We will say that such a near decagon is of *order* $(1, t; \mu, 1, 1)$. Conversely, each near octagon of order $(1, t; \mu, 1, 1)$ arises from a partial quadrangle of order (t, t, α).

Unfortunately, no thick symmetric partial quadrangle, which is not a generalized quadrangle, is known to exist.

Part I

The Displacements of Automorphisms of some finite Geometries

Introduction

Given a finite generalized quadrangle of order (s,t), and a collineation θ, there is a connection between the parameters s,t, the number f_0 of fixed points and the number f_1 of points mapped under θ to collinear points, given by Benson's theorem [5], see 0.2:

$$(1+t)f_0 + f_1 \equiv 1 + st \mod s+t.$$

The natural question arising here is whether there exists a similar formula for a *duality* of a generalized quadrangle $\mathcal{S}$ of order s. Of course, a duality cannot fix points or lines, but elements can be mapped to elements at distance 1 or 3, and we ask ourselves whether we can say more about the number of points mapped onto a line at distance 1 and 3, respectively. Also, more generally, one can ask for similar restrictions on collineations and dualities for an arbitrary finite generalized polygon, and other important classes of finite geometries such as partial geometries, symmetric designs, near polygons, partial quadrangles. This is exactly what we are going to do in Part I of our study.

There are some immediate remarkable observations to make, when we compare all the formulae that we will derive. One of the most eye-catching results is that, in the generic cases, a duality "often" seems to have exactly $t+1$ absolute points, where $t+1$ is also exactly the number of points on a line. This strange common behaviour of so many different geometries is hard to explain, as in each class there are exceptions to this rule: in general when the parameter t is a nice number, such as a square, or twice a square, etc., and also when the order of the duality is not coprime to a certain parameter, which is almost always a prime power in the known examples.

Regarding our methods, the extension of Benson's formula has a straightforward part, but there is also a less trivial observation which precisely allows us to draw some rather strong conclusions in the case of dualities, thereby producing new results even for finite projective planes. The key idea is to use powers of a certain matrix M, whereas in the proof of Benson's original formula, one only uses the matrix M itself. The key lemma in this connection is Lemma 1.2.5 below. We will need this lemma in each chapter of Part

I. Apart from that lemma, there is another difficulty that we must overcome. Indeed, Benson's original formula rests on the fact that the eigenvalues of an adjacency matrix of any thick generalized quadrangle are integers. This is no longer the case for the thin geometries that we need to use in order to produce formulae for dualities. Here, we use some number theoretic results to solve this problem, see Lemma 1.2.2. In general this enables us to write down some strong formulae, but in certain cases, this breaks down.

The ideas of the proofs of the basic formulae in the various chapters are the same, but the calculations are different, as the geometries behave differently with respect to distances of their elements. That is why we always carry out these calculations with great care. Also, we repeat in each chapter the basics of the method as we hope that this enables readers to skip chapters and go straight to their favourite geometries, be it partial geometries, partial quadrangles or symmetric designs. Only the initial, nontrivial but general part of the arguments is not repeated and is collected in Section 1.2 of Chapter 1.

Although some formulae might seem hopelessly complicated, they can be useful, as we will see in Part II. The results of Chapter 1 appeared in [44], up to a slight oversight that we correct in this thesis.

1

Generalized polygons

In this chapter we will generalize Benson's theorem to all finite generalized polygons. In particular, given a collineation θ of a finite generalized polygon $\mathcal{S}$, we obtain a relation between the parameters of $\mathcal{S}$ and, for various natural numbers i, the number of points x which are mapped by θ to a point at distance i from x. As a special case we consider generalized $2n$-gons of order $(1,t)$ and use these to determine, in the generic case, the exact number of absolute points of a given duality of the underlying generalized n-gon of order t. This produces new results, even for finite projective planes.

The main application of our results lies in the classification of finite generalized polygons whose collineation or duality group satisfies some given transitivity property like flag-transitivity, or sharp transitivity on points or lines. For an explicit application, see [41]. We will also apply our formulae in Part II of this thesis to prove the non-existence of certain collineations with prescribed displacement properties.

1.1 Notation and main results

Let θ be a collineation of a finite generalized n-gon $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s,t). Let f_i, $0 \leq i \leq \frac{n}{2}$, be the number of points x of $\mathcal{S}$ that are mapped under θ onto a point at distance

i from x, measured in the point graph (or, equivalently, distance $2i$ in the incidence graph). Also, the point graph has one eigenvalue (namely, $s(1+t)$) with multiplicity 1, it has always an eigenvalue equal to $-1-t$ and it has $\frac{n}{2}-1$ eigenvalues different from $-1-t$ with multiplicity greater than 1. We denote the latter with ξ_j, $1 \leq j \leq \frac{n}{2}-1$.

Benson's theorem [5] says that, if $n = 4$, then $(1+t)f_0 + f_1 \equiv 1 + st \mod s+t$. Equivalently, there exists an integer k_1 such that $(1+s)(1+t)+k_1(s+t) = (1+t)f_0+f_1$. The main result of this chapter generalizes this equivalent formulation as follows:

Main Result *With the above notation, for $n \in \{4, 6, 8, 12\}$, and under some mild restriction on the order of θ, there exist integers k_j, $1 \leq j \leq \frac{n}{2}-1$, and, for each $k \in \{1, 2, \ldots, \frac{n}{2}-1\}$, explicitly defined polynomial expressions $P_{k,i}(s,t)$, $0 \leq i \leq k-1$, in s and t, such that*

$$(1+s)^k(1+t)^k + \sum_{j=1}^{n/2-1} k_j(\xi_j+1+t)^k = f_k + \sum_{i=0}^{k-1} P_{k,i}(s,t)f_i.$$

We explicitly determine the expressions $P_{k,i}$ in this theorem below, for each n separately: for $n = 4$, see Theorem 1.3.1 (this is Benson's theorem); for $n = 6$, see Theorems 1.4.1 and 1.4.4; for $n = 8$, see Theorems 1.5.1, 1.5.4 and 1.5.5; for $n = 12$, see Theorems 1.6.1, 1.6.2, 1.6.3, 1.6.4 and 1.6.5. Also, the "mild restriction" mentioned above is each time explicitly included. The reason for that restriction is explained in the next section. That it is indeed "mild" is shown by the fact that it is void in the thick case.

Although the expressions $P_{k,i}$ are, especially for the cases $n = 8, 12$, rather involved and cumbersome, we are able to draw some interesting conclusions. In particular we obtain strong restrictions on the number of absolute elements of a duality. Generically, we obtain the following result.

Main Corollary *Let θ be a duality of order m of a finite generalized n-gon with parameter s. If m and s are relatively prime, s is not a square if $n = 3$, $2s$ is not a square if $n = 4$, and none of s and $3s$ are squares if $n = 6$, then there are exactly $1 + s$ absolute points for θ.*

Much more precise information is contained in the corollaries below (in particular the condition of m and s being relatively prime is a sufficient condition, which can be made much weaker, but more technical to state; for details, see below). The consequences of our Main Result seem endless, and we have included only a few of them. They are related to ovoids, subpolygons, involutions and dualities.

This chapter is organized as follows. In Section 1.2, we prove some general statements about eigenvalues and multiplicities, and recall a useful result concerning the adjacency matrix of a generalized polygon. In Section 1.3 we repeat Benson's theorem and write down some consequences, as a warming-up for the more involved cases treated in Section 1.4 (generalized hexagons), Section 1.5 (generalized octagons) and Section 1.6 (generalized dodecagons).

1.2 Some general observations

We will use the following notation. Suppose that $\mathcal{S}$ is a finite geometry (a generalized polygon in this chapter) of order (s,t). Let v be the number of points of $\mathcal{S}$ and b the number of lines. Put $\mathcal{P} = \{x_i : 1 \leq i \leq v\}$ and $\mathcal{L} = \{L_j : 1 \leq j \leq b\}$. Let D be an incidence matrix of $\mathcal{S}$, i.e., the rows of D are labelled by the points of $\mathcal{S}$, the columns by the lines of $\mathcal{S}$ and the (x,L)-entry of D (where $x \in \mathcal{P}$ and $L \in \mathcal{L}$) is equal to 1 if $x \mathtt{I} L$; otherwise it is 0. Then $M := DD^T = A + (t+1)I$, where A is an adjacency matrix of the point graph of $\mathcal{S}$. Let θ be an automorphism of $\mathcal{S}$ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise (in fact Q is the permutation matrix belonging to θ with respect to the action on $\mathcal{P}$). Similarly, let $R = (r_{ij})$ be the permutation matrix belonging to θ with respect to the action of θ on $\mathcal{L}$ (so $r_{ij} = 1$ if $L_i^\theta = L_j$ and $r_{ij} = 0$ otherwise). Then $DR = QD$. Because Q and R are permutation matrices, it follows that $Q^T = Q^{-1}$ and $R^T = R^{-1}$, so we have $QM = QDD^T = DRD^T = DRR^TD^T(Q^{-1})^T = DD^TQ = MQ$. Hence $QM = MQ$. Because n is the order of θ and $QM = MQ$, we have $(QM)^n = Q^nM^n = M^n$. It follows that the eigenvalues of QM are of the form $\xi\lambda$ with λ an eigenvalue of M and ξ an n^{th} root of unity. Note that the eigenvalue $(1+s)(1+t)$ of M is also an eigenvalue of QM with multiplicity 1.

So, we need to know something about the eigenvalues of QM. In Benson's original setting, this was possible because the eigenvalues of M are integers, and then we have the following easy lemma.

Lemma 1.2.1 *Suppose that ξ and ξ' are both primitive d^{th} roots of unity, with d a divisor of n, and let λ be an integer eigenvalue of M. If at least one of $\xi\lambda$ and $\xi'\lambda$ is an eigenvalue of QM, than they both are and they have the same multiplicity.*

Proof. The coefficients of the characteristic polynomial of QM are integers. The minimal polynomials (over $\mathbb{Q}$) of $\xi\lambda$ and $\xi'\lambda$ coincide, hence $\xi\lambda$ and $\xi'\lambda$ have the same multiplicity. $\square$

That this lemma fails for non-integer eigenvalues will be demonstrated in Chapter 4 with the only example of a geometry in the classes that we consider which is not the double of another geometry and which does not admit integer eigenvalues, namely, the ordinary pentagon.

In order to produce a similar lemma allowing non-integer eigenvalues, we first define compatibility. Let $\mathbb{Q}(e^{\frac{2i\pi}{n}})$ be the n^{th} cyclotomic extension of the rational numbers $\mathbb{Q}$. Then we say that a nonnegative integer m is *compatible* with the natural number n if $\sqrt{m} \notin \mathbb{Q}(e^{\frac{2i\pi}{n}}) \setminus \mathbb{Q}$. In particular, every nonnegative integer is compatible with 1 and 2 (trivially), but also with 3, 4 and 6. Here and below, i is a square root of -1 in the complex numbers $\mathbb{C}$.

It will turn out that the doubles of the geometries that we consider frequently do admit non-integer eigenvalues. But in these cases the point graph is bipartite and the eigenvalues of the matrix A are integers together with perfect square roots. Slightly more generally, we could consider geometries for which the point graph admits eigenvalues that involve only integers and square roots of positive integers, such as the dual of the double of thick generalized polygons, symmetric designs, partial geometries and partial quadrangles. Then we have the following lemma, which generalizes in a straightforward way Lemma 1.2.1 above.

Lemma 1.2.2 *Suppose that ξ and ξ' are both primitive d^{th} roots of unity, with d a divisor of n, and let $\lambda = a + b\sqrt{c}$ be an eigenvalue of M, with a, b integers, and c a (square-free) positive integer. Suppose c is compatible with d. If at least one of $\xi\lambda$ and $\xi'\lambda$ is an eigenvalue of QM, then they both are and they have the same multiplicity.*

Proof. The Galois group of the extension $[\mathbb{Q}(e^{\frac{2i\pi}{d}}, \sqrt{c}) : \mathbb{Q}(\sqrt{c})]$ leaves the characteristic polynomial of QM invariant. Moreover, this group fixes $\sqrt{c}$ and acts transitively on the roots of the d^{th} cyclotomic polynomial. This implies that the eigenvalue $\lambda\xi$ has the same multiplicity as the eigenvalue $\lambda\xi'$. $\square$

So in order to be able to apply the previous lemma in concrete situations, we need to know which natural numbers are compatible with which natural numbers. Although in this thesis we will only meet situations, where, with the notation of Lemma 1.2.2, c is a prime or the product of a prime with 2 or 3, we can answer this question in full generality (this is in fact well-known, but we were not able to find a precise reference). First we treat prime numbers.

Proposition 1.2.3 *Let p be a prime and n a natural number. Then p is compatible with n if and only if exactly one of the following cases occurs.*

(1) p does not divide n;

(2) $p = 2$, n is even and 8 does not divide n;

(3) $p \equiv 3 \mod 4$, p divides n and 4 does not divide n.

In particular, all primes are compatible with 2, 3, 4 and 6.

Proof. We start by remarking that, as soon as p does not divide n, then p does not ramify in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, see Proposition 2.3 in [59], and hence p cannot be a square in that field. We are left with the cases where p does divide n.

Suppose p is odd and divides n. The Gaussian sum

$$\sqrt{\left(\frac{-1}{p}\right)p} = \sum_{\ell=1}^{p-1} \left(\frac{\ell}{p}\right) e^{2i\pi\ell/p},$$

where $\left(\frac{a}{p}\right)$ is the Lengendre symbol (equal to 1 if a is a square mod p, and -1 otherwise), tells us already that $\sqrt{p}$ belongs to $\mathbb{Q}(e^{\frac{2i\pi}{p}})$, and hence to $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, if $p \equiv 1 \mod 4$. If $p \equiv 3 \mod 4$, then the same formula shows that $\sqrt{-p}$ belongs to $\mathbb{Q}(e^{\frac{2i\pi}{p}})$. But if $\sqrt{-p}$ belongs to this field, then $\sqrt{p}$ also belongs to it if and only if i belongs to it. Hence $\sqrt{p}$ belongs to $\mathbb{Q}(e^{\frac{2i\pi}{n}})$ if and only if $4p$ divides n.

If $p = 2$, then clearly $\sqrt{2} \in \mathbb{Q}(e^{\frac{2i\pi}{8}}) = \mathbb{Q}(e^{\frac{2i\pi}{4}}, \sqrt{2})$ and $\sqrt{2} \notin \mathbb{Q}(e^{\frac{2i\pi}{4}})$ imply the result. $\square$

We can now give a general answer to the question of compatibility. Obviously, we may assume that c is square-free, hence the product of distinct primes.

Proposition 1.2.4 *Let c be the product of k distinct primes and n a natural number. Then c is compatible with n if and only if exactly one of the following cases occurs.*

(1) c does not divide n;

(2) c is even and divides n, and 8 does not divide n;

(3) $c \equiv 3 \mod 4$, c divides n and 4 does not divide n.

In particular, all natural numbers are compatible with 2, 3, 4 and 6.

Proof. First we remark that, if c does not divide n, then m is compatible with n. Indeed, suppose $\sqrt{c}$ belongs to $\mathbb{Q}(e^{\frac{2i\pi}{n}})$. Since in the ring of algebraic integers, the ideals generated by two distinct primes of $\mathbb{Z}$ have no common prime factor (this follows easily from the fact that each $\mathbb{Z}$-prime ideal is the intersection of $\mathbb{Z}$ with any prime ideal factor in the ring of algebraic integers), and since at least one prime factor of c does not ramify in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, we obtain a contradiction.

Now suppose that c divides n. If $c \equiv 1 \mod 4$, then the number of primes congruent to 3 modulo 4 dividing c is even, and so, since for each such prime number p we have $\sqrt{-p} \in \mathbb{Q}(e^{\frac{2i\pi}{n}})$ by the previous proposition, we see, taking the product of all such expressions, and taking Lemma 1.2.3 into account for primes congruent to 1 modulo 4, that $\sqrt{c} \in \mathbb{Q}(e^{\frac{2i\pi}{n}})$. This completes the proof for c congruent to 1 modulo 4.

If $c \equiv 3 \mod 4$, then, the number of prime numbers dividing c and congruent to 3 modulo 4 is odd. Let p be one of them. Then, by the above, c/p is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$. Hence, if c is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, then p is and the result follows from the previous proposition. Conversely, if 4 divides n, then p is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, and so is c. This shows (3).

Finally, let c be even. If 8 divides n, then by the previous proposition, 2 is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$ and by the foregoing paragraphs $c/2$ is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, hence c is. Conversely, suppose c is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$. If $c/2$ is 1 modulo 4, then $c/2$ is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, and so 2 must be. Then the result follows from Proposition 1.2.3(2). If $c/2$ is 3 modulo 4, then let p be a prime dividing c with $p \equiv 3 \mod 4$. Then $c/(2p)$ is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, hence $2p$ is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$. Since $-p$ is a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$, we deduce that -2 must be a square in $\mathbb{Q}(e^{\frac{2i\pi}{n}})$. This is only the case if 8 divides n, since i belongs to $\mathbb{Q}(e^{\frac{2i\pi}{n}})$ if and only if 4 divides n.

This completes the proof of the proposition. □

From now on, i is not anymore $\sqrt{-1}$.

Our last general lemma concerns the connection between the multiplicity of an eigenvalue of QM and the multiplicity of the corresponding eigenvalue of QM^j, with j a positive integer.

Lemma 1.2.5 *There exists a common orthonormal basis of eigenvectors of Q and M such that, if $\vec{v}$ is such a basis vector with eigenvalue ξ for Q and λ for M, then $\vec{v}$ is an eigenvector of QM with eigenvalue $\xi\lambda$, and $\vec{v}$ is also an eigenvector of QM^j with eigenvalue $\xi\lambda^j$.*

Proof. Let λ be an eigenvalue of M and let V be the corresponding eigenspace. Note that, if $\vec{v} \in V$, then

$$\begin{aligned} MQ\vec{v} &= QM\vec{v} \\ &= Q\lambda\vec{v} \\ &= \lambda Q\vec{v}, \end{aligned}$$

and hence $Q\vec{v} \in V$. So Q preserves V and induces an isometry in V, V viewed as the standard Euclidean space. Hence there exists an orthonormal basis B of V of eigenvectors of Q. Since $Q^n = 1$, each eigenvalue ξ is an n^{th} root of unity. If $\vec{v} \in B$ is an eigenvector for Q with eigenvalue ξ, then $MQ\vec{v} = \xi\lambda\vec{v}$. Hence $\vec{v}$ is an eigenvector of QM with eigenvalue $\xi\lambda$. We also compute

$$\begin{aligned} QM^j\vec{v} &= M^{j-1}(QM\vec{v}) \\ &= M^{j-1}\xi\lambda\vec{v} \\ &= \xi\lambda M^{j-1}\vec{v} \\ &= \xi\lambda^j\vec{v}. \end{aligned}$$

Hence $\vec{v}$ is an eigenvector of QM^j with eigenvalue $\xi\lambda^j$. The assertion now follows from the fact that M is diagonalizable. $\square$

From now on, we again specialize to generalized polygons. Note that, in the sequel, we will also consider *thin* generalized polygons, by which we mean generalized polygons $\mathcal{S}$ of order (s,t) with either $s = 1$ or $t = 1$. As already noted, a generalized $2n$-gon $\mathcal{S}$ of order $(1,t)$ is the double $2\mathcal{S}'$ of a — up to duality uniquely defined — generalized n-gon $\mathcal{S}'$ of order t. Every collineation of $\mathcal{S}$ induces either a unique collineation of $\mathcal{S}'$ or a unique duality of $\mathcal{S}'$. In the first case a point x of $\mathcal{S}$ is mapped onto a point at even distance from x; in the second case a point x of $\mathcal{S}$ is mapped onto a point at odd distance from x. In the sequel we will call $\mathcal{S}'$ the *underlying generalized n-gon.*

The following lemma is well known, e.g. see [8].

Lemma 1.2.6 *Let A be the adjacency matrix of a generalized $2n$-gon of order (s,t). Since for two points x and y, the (x,y)-entry only depends on the distance $2i$ (in the incidence graph) between x and y, the same is true for A^k with $k \geq 1$. Hence we can denote this entry by $a_i^{(k)}$. Let p_j^i be the number of points at distance $2j$ from x and collinear to y, with x and y as above. Then $a_i^{(k+1)} = \sum_{j=0}^{n} p_j^i a_j^{(k)}$, with $i \in \{0,1,\cdots,n\}$.*

For a generalized n-gon p^i_j is equal to:

$$\begin{cases} s(t+1) & \text{if } j=1, i=0 \\ 1 & \text{if } 0 \leq j = i-1 \leq n-2 \\ s-1 & \text{if } 1 \leq j = i \leq n-1 \\ st & \text{if } 2 \leq j = i+1 \leq n \\ t+1 & \text{if } j = n-1, i = n \\ (s-1)(1+t) & \text{if } j = i = n \\ 0 & \text{otherwise.} \end{cases}$$

We end with a straightforward observation, valid for all geometries.

Lemma 1.2.7 *A duality θ of a point-line geometry has as many absolute points as absolute lines.*

Proof. This follows from the fact that, if $x \mathtt{I} x^\theta$, then $x^\theta \mathtt{I} (x^\theta)^\theta$, which implies that, if x is an absolute point, then x^θ is an absolute line. It is now easy to see that θ induces a bijection from the set of absolute points to the set of absolute lines. □

1.3 Collineations of generalized quadrangles

At first we will have a look at a generalized quadrangle $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ of order (s,t). Let D, M, A, Q and θ be defined as before. If $M = A + (t+1)I$, then M has eigenvalues $\tau_0 = (1+s)(1+t)$, $\tau_1 = 0$ and $\tau_2 = s+t$, with respective multiplicities $m_0 = 1$, $m_1 = s^2(1+st)/(s+t)$ and $m_2 = st(1+s)(1+t)/(s+t)$ (cf. Table 6.4 in [8]). Now we have the following theorem.

Theorem 1.3.1 (Benson [5]) *If f_0 is the number of points fixed by the automorphism θ and if f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then*

$$\mathrm{tr}(QM) = (1+t)f_0 + f_1 \quad \textit{and} \quad (1+t)f_0 + f_1 \equiv 1 + st \mod s+t.$$

Proof. For the proof of this theorem, we refer to [5]. □

Remark 1.3.2 We can also write the conclusion of Theorem 1.3.1 as follows:

$$\mathrm{tr}(QM) = (1+t)f_0 + f_1 = k(s+t) + (1+s)(1+t).$$

We now collect some consequences of Theorem 1.3.1. We do not claim originality, but they are similar to some results that we will prove for hexagons and hence we include them for completeness' sake. (e.g. Corollary 1.3.5 has been proved by K. Thas (personal communication)).

Corollary 1.3.3 *Let $\mathcal{S}$ be a generalized quadrangle of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. Suppose that there are no fixpoints and no points which are mapped to a collinear point. Then $f_0 = f_1 = 0$. Because of the previous theorem $1+st+l(s+t)$ has to be 0 for some l, or $st + l(s+t) = -1$. But because s and t are not relatively prime, there exists an integer $m > 1$ which divides both s and t. Hence m divides $st + l(s+t)$. But m does not divide -1 and we have a contradiction. □

Corollary 1.3.4 *Suppose that $\mathcal{S}$ is a generalized quadrangle of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is even, then θ cannot fix any ovoid pointwise.*

Proof. Suppose that $\mathcal{O}$ is an ovoid which is fixed pointwise by θ. Then (with the notation of the previous theorem) $f_0 = 1+s^2 = |\mathcal{O}|$. Note that by 2.4.1 in [38] θ cannot fix anything else. Suppose that there exists a point x which is mapped to a collinear point by θ. Take a line through x different from xx^θ. This line contains a point y of the ovoid. Now the line xy is mapped to the line $x^\theta y$, hence we have a triangle which is a contradiction. So $f_1 = 0$. Because of Benson's theorem it follows that $(1+s)(1+s^2) \equiv 1+s^2 \mod 2s$ or $s(1+s^2) \equiv 0 \mod 2s$. Hence $1+s^2$ is even and so s has to be odd. □

Let $\mathsf{Q}(4,q)$ be a nonsingular (parabolic) quadric in the projective space $\mathsf{PG}(4,q)$, and let H be a hyperplane of $\mathsf{PG}(4,q)$ meeting $\mathsf{Q}(4,q)$ in a nonsingular elliptic quadric $\mathsf{Q}^-(3,q)$. If q is odd, then H has a pole x (the intersection of all hyperplanes tangent to $\mathsf{Q}(4,q)$ at points of $\mathsf{Q}^-(3,q)$), and the unique involutive perspectivity of $\mathsf{PG}(4,q)$ with center x and axis H induces a nontrivial collineation in the generalized quadrangle $\mathsf{Q}(4,q)$ fixing the ovoid $\mathsf{Q}^-(3,q)$ pointwise. Hence Corollary 1.3.4 is not valid for s odd.

Corollary 1.3.5 *Suppose that $\mathcal{S}$ is a generalized quadrangle of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is even, then θ cannot fix any thin subquadrangle of order $(1,s)$ pointwise.*

Proof. Suppose that $\mathcal{S}'$ is a thin subquadrangle of $\mathcal{S}$ of order $(1, s)$ which is fixed pointwise by θ. Then we claim $f_0 = 2(s+1)$, which is the number of points of $\mathcal{S}'$. Indeed, if θ fixes a point not belonging to $\mathcal{S}'$, then θ would fix a subquadrangle $\mathcal{S}''$ of order (s'', s) with $\mathcal{S}' \subset \mathcal{S}'' \subseteq \mathcal{S}$. By 2.2.1 of [38] it follows that $s \geq s''s$. Hence $s'' = 1$ a contradiction. Now suppose that x is a point of $\mathcal{S}'$. There are $s+1$ lines through x in $\mathcal{S}'$ and on each of these lines there is precisely one other point which belongs to $\mathcal{S}'$. Take such a point y. On the line xy of $\mathcal{S}$ there are $s-1$ other points which are mapped to each other (not fixed). So we already have $(s-1)(s+1)^2$ points which are mapped to a collinear point by θ. Suppose that there is a point z of $\mathcal{S}$ which does not lie on a line of $\mathcal{S}'$ and which is mapped to a point collinear to itself. Then by 2.2.1 of [38] the line zz^θ intersects a line of $\mathcal{S}'$ (which is fixed under θ). But then the line zz^θ also has to be fixed and we obtain a contradiction. Hence $f_1 = (s-1)(s+1)^2$. Now Theorem 1.3.1 implies that there exists an integer k with

$$k(2s) + (1+s)^2 = 2(1+s)^2 + (s-1)(s+1)^2.$$

Hence $k = \frac{(s+1)^2}{2}$ and so s is odd. □

An example that this corollary is not valid for s odd can be obtained similarly as the example illustrating that Corollary 1.3.4 fails for s odd, by considering a hyperplane meeting $\mathsf{Q}(4, q)$ in a nonsingular hyperbolic quadric $\mathsf{Q}^+(3, q)$.

1.4 Collineations of hexagons and dualities of projective planes

Next we will generalize Benson's theorem for hexagons.

Suppose that $\mathcal{S}$ is a generalized hexagon. Let θ be an automorphism of $\mathcal{S}$ and let f_0 be the number of fixpoints, f_1 the number of points which are mapped to a collinear point ($\mathrm{d}(x^\theta, x) = 1$ in the point graph of $\mathcal{S}$) and f_2 the number of points which are mapped to a point at distance 2 from itself (in the point graph). The matrix M is again equal to $A + (t+1)I$, with A an adjacency matrix of the point graph of $\mathcal{S}$. And Q is the matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise.

We start from the eigenvalues of A which are $-1-t$, $s(t+1)$, $-1+s+\sqrt{st}$ and $-1+s-\sqrt{st}$, with respective multiplicities m_0, $m_1 = 1$, m_2 and m_3. Because $M = A + (t+1)I$ the eigenvalues of M are as follows (cf. Table 6.4 in [8]).

eigenvalues of M	multiplicity
0	$m_0 = \frac{s^3(1+st+s^2t^2)}{s^2+st+t^2}$
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{st}$	$m_2 = \frac{(1+t)st(1+s)(1+st+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+3st+(s-1)(t-1)\sqrt{st})}$
$s+t-\sqrt{st}$	$m_3 = \frac{(1+t)st(1+s)(1+st+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+3st-(s-1)(t-1)\sqrt{st})}$

We now have the following result.

Theorem 1.4.1 *Let $\mathcal{S}$ be a generalized hexagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that st is compatible with the order of θ (which is automatic when $\mathcal{S}$ is thick). If f_i, $i = 0, 1$, is defined as above, then for some integers k_1 and k_2 there holds*

$$k_1(s+t+\sqrt{st}) + k_2(s+t-\sqrt{st}) + (1+s)(1+t) = (1+t)f_0 + f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^nM^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (1+s)(1+t)J$, we have $(QM)J = (1+s)(1+t)J$, so $(1+s)(1+t)$ is an eigenvalue of QM. Because $m_1 = 1$, it follows that this eigenvalue of QM has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM with multiplicity m_0. For each divisor d of n, let ξ_d denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Now U_d is the coefficient of the term of the second largest degree of the corresponding cyclotomic polynomial $\Phi_n(x)$. And since $\Phi_n(x) \in \mathbb{Z}[x]$, by [22], U_d is an integer. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = s+t+\sqrt{st}$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM with $|\varphi'| = s+t-\sqrt{st}$, because of Lemmas 1.2.1 and 1.2.2 and our assumption on compatibility. Let a_d denote the multiplicity of $\xi_d(s+t+\sqrt{st})$ and let b_d denote the multiplicity of $\xi_d(s+t-\sqrt{st})$ as eigenvalues of QM, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\mathrm{tr}(QM) = \sum_{d|n} a_d(s+t+\sqrt{st})U_d + \sum_{d|n} b_d(s+t-\sqrt{st})U_d + (1+s)(1+t),$$

or

$$\mathrm{tr}(QM) = k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + (1 + s)(1 + t),$$

with k_1 and k_2 integers.

Since the entry on the i^{th} row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\mathrm{tr}(QM) = (1 + t)f_0 + f_1$. Hence

$$k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + (1 + s)(1 + t) = (1 + t)f_0 + f_1,$$

with k_1 and k_2 integers. □

The following corollary is the analogue of Corollary 1.3.3

Corollary 1.4.2 *Let $\mathcal{S}$ be a thick generalized hexagon of order (s, t) and let θ be an automorphism of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. Suppose that there are no fixpoints and no points which are mapped to a collinear point. Then $f_0 = f_1 = 0$. Because of the previous theorem, $k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+(1+s)(1+t)$ has to be equal to 0. Hence $k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+s+t+st = -1$. But because s and t are not relatively prime, there exists an integer $m > 1$ which divides s and t. Hence m divides $k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st}) + s + t + st$, but m does not divide -1 and we have a contradiction. □

Corollary 1.4.3 *Let $\mathcal{S}$ be a thick generalized hexagon of order (s, t) and let θ be an involution of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one fixline.*

Proof. This follows immediately from the previous corollary because if there is a point x which is mapped to a point collinear to x by the involution θ, then the line xx^θ is a fixline. □

Now we have a look at the formula in Theorem 1.4.1 in the special case of a thin hexagon ($s = 1$). Then we have, assuming all prime divisors of t are compatible with the order of θ if t is not a perfect square:

$$k_1(1 + t + \sqrt{t}) + k_2(1 + t - \sqrt{t}) + 2(1 + t) = (1 + t)f_0 + f_1.$$

If t is not a square, then it follows that $k_1 = k_2$, hence we obtain $k_1 2(t+1) + 2(t+1) = (t+1)f_0 + f_1$ and so $t+1$ has to divide f_1. But we will improve this below (see Corollary 1.4.6). Note that either f_0 or f_1 is 0, according to whether the corresponding collineation θ of $\mathcal{S}$ induces a duality or a collineation in the underlying projective plane.

The method exploited in the proof of Theorem 1.4.1 is completely similar to the original approach of Benson. However, in order to be able to say more, an additional idea is needed. Motivated by Lemma 1.2.5, our idea is now to apply Benson's approach to the matrix M^2 (which, in case of generalized quadrangles, does not give anything new). Without Lemma 1.2.5, this would not give too much, but in combination with that lemma, we will obtain new and quite interesting results, even for the case of projective planes!

Theorem 1.4.4 *Let $\mathcal{S}$ be a generalized hexagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that st is compatible with the order of θ (which is automatic when $\mathcal{S}$ is thick). If f_i, $i = 0,1,2$, is as before, then for the integers k_1 and k_2 of* Theorem 1.4.1 *there holds*

$$k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + ((1+s)(1+t))^2 = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2.$$

Proof. Suppose that M, A and Q are defined as before. Suppose that θ has order n, so that $(QM^2)^n = Q^n M^{2n} = M^{2n}$. It follows that the eigenvalues of QM^2 are the eigenvalues of M^2 multiplied by the appropriate roots of unity. Since $M^2 J = ((1+s)(1+t))^2 J$, we have $(QM^2)J = ((1+s)(1+t))^2 J$, so $((1+s)(1+t))^2$ is an eigenvalue of QM^2. By Lemma 1.2.5 and the proof of Theorem 1.4.1 it follows that this eigenvalue of QM^2 has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM^2 with multiplicity m_0. For each divisor d of n, let ξ_d again denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1,2,\dots,d-1\}$ that are relatively prime to d. Then, as above, U_d is an integer. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM^2 with $|\varphi| = (s+t+\sqrt{st})^2$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM^2 with $|\varphi'| = (s+t-\sqrt{st})^2$, because of Lemmas 1.2.1 and 1.2.2. Let a_d denote the multiplicity of $\xi_d(s+t+\sqrt{st})^2$ and let b_d denote the multiplicity of $\xi_d(s+t-\sqrt{st})^2$ as eigenvalues of QM^2, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\mathrm{tr}(QM^2) = \sum_{d|n} a_d(s+t+\sqrt{st})^2 U_d + \sum_{d|n} b_d(s+t-\sqrt{st})^2 U_d + ((1+s)(1+t))^2,$$

or

$$\mathrm{tr}(QM^2) = k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + ((1+s)(1+t))^2,$$

with k_1 and k_2 integers. Clearly we have $\mathrm{tr}(QA) = f_1$.

The matrix $A^2 = (a_{ij})$ is the matrix with $s(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = s-1$ if $x_i \sim x_j$, $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 2$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^2) = s(1+t)f_0 + (s-1)f_1 + f_2$. It follows that

$$\begin{aligned}
& \mathrm{tr}(QM^2) \\
= \; & \mathrm{tr}(Q(A+(1+t)I)^2) \\
= \; & \mathrm{tr}(QA^2) + 2(1+t)\mathrm{tr}(QA) + (1+t)^2\mathrm{tr}(Q) \\
= \; & s(1+t)f_0 + (s-1)f_1 + f_2 + 2(1+t)f_1 + (1+t)^2 f_0 \\
= \; & (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2.
\end{aligned}$$

Finally, the integers k_1 and k_2 are the same integers as in Theorem 1.4.1 by Lemma 1.2.5.

This completes the proof of the theorem. □

Corollary 1.4.5 *Suppose that we have a thin hexagon of order* $(1,t)$, *with* $t \neq 1$. *Consider a duality* θ *in the underlying projective plane. Suppose that* t *is compatible with the order of* θ. *If* t *is not a square, then* $f_1 = 2(1+t)$. *If* t *is a square, then* $f_1 \equiv 2 \mod 2\sqrt{t}$. *In particular there is at least one absolute line and one absolute point.*

Proof. Since we have a duality in the underlying projective plane, we know that $f_0 = 0$ and $f_2 = 0$. Because of Theorems 1.4.1 and 1.4.4, we have the following equations:

$$\begin{cases} k_1(1+t+\sqrt{t}) + k_2(1+t-\sqrt{t}) + 2(1+t) = f_1, \\ k_1(1+t+\sqrt{t})^2 + k_2(1+t-\sqrt{t})^2 + (2(1+t))^2 = (2+2t)f_1, \end{cases}$$

hence

$$\begin{cases} k_1 = \frac{f_1 - 2(1+t)}{2\sqrt{t}}, \\ k_2 = -\frac{f_1 - 2(1+t)}{2\sqrt{t}}. \end{cases}$$

So $\frac{f_1-2(1+t)}{2\sqrt{t}}$ has to be an integer. In the case that t is not a square, this only holds if $f_1 - 2(1+t) = 0$. Hence $f_1 = 2(1+t)$ if t is not a square. If t is a square, then $f_1 - 2$ has to be a multiple of $2\sqrt{t}$. Hence $f_1 \equiv 2 \mod 2\sqrt{t}$. □

In view of Lemma 1.2.7, this immediately implies:

Corollary 1.4.6 *Suppose that θ is a duality of a projective plane of order t. Suppose that t is compatible with the order of θ. Then θ has $1+t$ absolute points and $1+t$ absolute lines if t is not a perfect square, and it has $1 \mod \sqrt{t}$ absolute points and just as many absolute lines if t is a perfect square.*

This corollary is well known for polarities, see Lemma 12.3 in Hughes & Piper [29]. By Proposition 1.2.3, this now also holds for every duality of order 4 or 6 in any finite projective plane. If the plane has prime power order, then we essentially only have to make possible exceptions for dualities whose order is a multiple of that prime.

If t is not compatible with the order of θ (and hence t is not a perfect square), then the conclusion of the Corollary is not necessarily valid anymore. Not only because our proof fails, but also because of the following simple counter example. Let $t = 2$ and consider the duality of $\mathsf{PG}(2,2)$ corresponding to the following matrix:

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

One easily computes that θ has order 8. Hence, since 2 is not compatible with 8, we cannot deduce from the corollary that there must be exactly 3 absolute points. In fact one calculates that (x, y, z) are the coordinates of an absolute point if and only if $(x+y)y = 0$, implying that there are five absolute points.

Interesting to note is that there is also a duality θ' of order 8 which admits only 1 absolute point, namely the one corresponding with the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

Notice also that $\theta^2 = \theta'^2$ and that, up to conjugacy, there are no other elements of order 8 .

If t is a square, then the lower bound of Corollary 1.4.6 can be obtained. Indeed, let b be an element of the finite (Galois) field $\mathbb{F}_t$ (of t elements) not belonging to the subfield $\mathbb{F}_{\sqrt{t}}$. With the usual representation of the Desarguesian projective plane $\mathsf{PG}(2,t)$ of order t by means of triples of elements of $\mathbb{F}_t$ (with round brackets to denote points and square brackets for lines), the map

$$\theta : \mathsf{PG}(2,t) \to \mathsf{PG}(2,t) : \begin{cases} (x,y,z) & \mapsto & [x^{\sqrt{t}}, z^{\sqrt{t}}, y^{\sqrt{t}} - bz^{\sqrt{t}}], \\ [u,v,w] & \mapsto & (u^{\sqrt{t}}, w^{\sqrt{t}} + bv^{\sqrt{t}}, v^{\sqrt{t}}) \end{cases}$$

is a duality. A point (x,y,z) is absolute if and only if (x,y,z) is incident with $[x^{\sqrt{t}}, z^{\sqrt{t}}, y^{\sqrt{t}} - bz^{\sqrt{t}}]$. If $z \neq 0$, then this obviously implies that

$$b = \frac{xx^{\sqrt{t}} + (yz^{\sqrt{t}} + y^{\sqrt{t}}z)}{zz^{\sqrt{t}}}$$

is fixed by the field automorphism $a \mapsto a^{\sqrt{t}}$, and hence belongs to $\mathbb{F}_{\sqrt{t}}$, a contradiction. Hence, if (x,y,z) is absolute, then $z = 0$ and so $xx^{\sqrt{t}} = 0$. We obtain a unique absolute point $(0,1,0)$. Likewise, $[0,0,1]$ is the unique absolute line.

For the next corollary we need the following lemma.

Note that for a generalized hexagon of order (s,t) with $s \neq t$ no ovoids exist because of [34].

Lemma 1.4.7 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is a collineation of $\mathcal{S}$. If $\mathcal{O}$ is an ovoid of $\mathcal{S}$ which is fixed pointwise by θ and θ additionally fixes some point $x \notin \mathcal{O}$, then θ is the identity.*

Proof. There exists a unique point of the ovoid which is collinear to x, we call this point y. Every point x' collinear to x which is not incident with the line xy is collinear to a unique point $y' \neq y$ of the ovoid. Since both x and y' are fixed, also x' is fixed. Now take any point $x'' \notin \{x,y\}$ that is incident with the line xy. Take a point a collinear to x'', not incident with xy. This point is collinear to a unique point b of $\mathcal{O}$. Since b is fixed by θ and the line xy is fixed by θ, also the point x'' has to be fixed. So every point collinear to x is fixed. Since $\mathcal{O}$ contains a point at distance 6 from x we can apply Theorem 4.4.2(v) in [55] to obtain that θ is the identity. ☐

It is well known that the dual $\mathsf{H}(q)^{\mathrm{dual}}$ of the split Cayley hexagon $\mathsf{H}(q)$ of order q admits an ovoid stabilized by the subgroup $\mathsf{SU}_3(q)$ of $\mathsf{G}_2(q)$, and the elements of $\mathsf{SU}_3(q)$ fixing the ovoid pointwise are exactly the elements of the center of $\mathsf{SU}_3(q)$ (see [12]). If q is divisible by 3, however, this center is trivial and the ovoid does not admit a nontrivial collineation fixing it pointwise. This is a special case of the following more general phenomenon.

Corollary 1.4.8 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is a multiple of 3, then θ cannot fix any ovoid pointwise.*

Proof. Suppose that $\mathcal{O}$ is an ovoid of $\mathcal{S}$ which is fixed pointwise by θ. Because of the previous lemma there are no other fixpoints. Then $f_0 = 1 + s^3 = |\mathcal{O}|$. Suppose that there exists a point x which is mapped to a collinear point by θ. Suppose that there is a point y of $\mathcal{O}$ which is collinear to x and not collinear to x^θ. Then the line xy will be mapped to the line $x^\theta y$, hence we have a triangle, so this is not possible. Now there has to be a point y' at distance 4 from x which belongs to the ovoid. By a similar reasoning we obtain a pentagon. Hence we have a contradiction and f_1 has to be equal to 0. Because θ fixes an ovoid every point is mapped to a point at distance at most 4 from it. So f_2 is the number of points not on the ovoid, hence $f_2 = (1 + s^3)(s + s^2)$. Because of Theorem 1.4.1 and Theorem 1.4.4 we have the following equations

$$\begin{cases} k_1 3s + k_2 s + (1+s)^2 = (1+s)(1+s^3), \\ k_1 9s^2 + k_2 s^2 + (1+s)^4 = (1+2s)(1+s)(1+s^3) + (1+s^3)(s+s^2). \end{cases}$$

Hence

$$\begin{cases} k_1 = \frac{(1+s)(s^2-1)}{3}, \\ k_2 = 0. \end{cases}$$

Because k_1 has to be an integer, it follows that s cannot be a multiple of 3. $\square$

The split Cayley hexagon $\mathsf{H}(q)$ admits a subhexagon of order $(1, q)$, stabilized by the group $\mathsf{SL}_3(q)$ (which has index two in the full stabilizer, see e.g. [17]). The pointwise stabilizer of that subhexagon is the center of $\mathsf{SL}_3(q)$, which is again trivial if 3 divides q. More generally, we can now show the following result.

Corollary 1.4.9 *Suppose that $\mathcal{S}$ is a generalized hexagon of order s and θ is a nontrivial automorphism of $\mathcal{S}$. If s is a multiple of 3, then θ cannot fix any thin subhexagon of order $(1, s)$ pointwise.*

Proof. Suppose that $\mathcal{S}'$ is a thin subhexagon of $\mathcal{S}$ of order $(1, s)$ which is fixed pointwise by θ. Then we claim $f_0 = 2(1 + s + s^2)$, which is the number of points of $\mathcal{S}'$. Indeed, if θ fixes a point not belonging to $\mathcal{S}'$, then θ would fix a subhexagon $\mathcal{S}''$ of order (s'', s) with $\mathcal{S}' \subset \mathcal{S}'' \subseteq \mathcal{S}$. By [48] it follows that $s \geq s''^2 t$ so $s \geq s''^2 s$. Hence $s'' = 1$, a contradiction. Now suppose that x is a point of $\mathcal{S}'$. There are $s + 1$ lines through x in $\mathcal{S}'$ and on each of these lines there is precisely one other point which belongs to $\mathcal{S}'$. Take such a point y. On the line xy of $\mathcal{S}$ there are $s - 1$ other points which are mapped to each other (not fixed). So we already have $(1+s+s^2)(s+1)(s-1)$ points which are mapped to a collinear point by θ. Suppose that there is a point z of $\mathcal{S}$ which does not lie on a line of $\mathcal{S}'$ and which is mapped to a point collinear to itself. Then by [48] the line zz^θ intersects a line of $\mathcal{S}'$ which is fixed under θ. But then the line zz^θ also has to be fixed which leads to a contradiction. Hence $f_1 = (1+s+s^2)(s+1)(s-1)$. Now suppose that there is a point u which is mapped to a point at distance 4 from itself. There exists a point u' which is collinear to u and also to u^θ. We have two possibilities, either the point u' is incident with a line L of $\mathcal{S}'$ or it is not. In the first case the point u' should be fixed, because it is the unique point from L collinear to both u and u^θ. But it cannot be a point of $\mathcal{S}'$ so we obtain a contradiction. In the second case, again by [48] the line uu' intersects a line M of $\mathcal{S}'$, so the line uu' is mapped to a line trough u^θ which intersects the same line M of $\mathcal{S}'$. Hence we obtain a quadrangle and we have a contradiction. Consequently $f_2 = 0$. Because of Theorem 1.4.1 and Theorem 1.4.4 we have the following equations

$$\begin{cases} k_1 3s + k_2 s + (1+s)^2 = (1+s)2(1+s+s^2) + (1+s+s^2)(1+s)(s-1), \\ k_1 9s^2 + k_2 s^2 + (1+s)^4 = (1+2s)(1+s)2(1+s+s^2) + \\ \quad (1+3s)(1+s+s^2)(1+s)(s-1). \end{cases}$$

Hence

$$\begin{cases} k_1 = \frac{(1+s)(1+s+s^2)}{3}, \\ k_2 = s + s^2. \end{cases}$$

We see that k_2 is an integer, but k_1 also has to be an integer. It follows that s cannot be a multiple of 3.

□

1.5 Collineations of octagons and dualities of quadrangles

Suppose that $\mathcal{S}$ is a generalized octagon. Let θ be an automorphism of $\mathcal{S}$ and let f_i be the number of points for which $\mathrm{d}(x, x^\theta) = i$ in the point graph. The matrices M, A and Q are defined analogously as before. The eigenvalues of M are as follows (cf. Table 6.4 in [8]):

eigenvalues of M	multiplicity
0	$m_0 = \frac{s^4(1+st)(1+s^2t^2)}{(s+t)(s^2+t^2)}$
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{2st}$	$m_2 = \frac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{4(s(t-1)^2+t(s-1)^2+2st+(s-1)(t-1)\sqrt{2st})}$
$s+t-\sqrt{2st}$	$m_3 = \frac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{4(s(t-1)^2+t(s-1)^2+2st-(s-1)(t-1)\sqrt{2st})}$
$s+t$	$m_4 = \frac{(1+t)st(1+s)(1+st)(1+s^2t^2)}{2(s(t-1)^2+t(s-1)^2+4st)}$

Theorem 1.5.1 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that $2st$ is compatible with the order of θ (which is automatic when $\mathcal{S}$ is thick). If f_0 and f_1 are as before, then for some integers k_1, k_2 and k_3 there holds*

$$k_1(s+t+\sqrt{2st}) + k_2(s+t-\sqrt{2st}) + k_3(s+t) + (1+s)(1+t) = (1+t)f_0 + f_1.$$

Proof. This proof is totally analogous to the proof of Theorem 1.4.1. □

Corollary 1.5.2 *Let $\mathcal{S}$ be a thick generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. If s and t are not relatively prime, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. This proof is totally analogous to the proof of Corollary 1.4.2. □

Note that in a thick generalized octagon s and t cannot be odd at the same time since $\sqrt{2st}$ is an integer and hence either the number of points or the number of lines is odd, or both are. So for a thick generalized octagon, we conclude:

Corollary 1.5.3 *Let $\mathcal{S}$ be a thick generalized octagon of order (s,t) and let θ be an involution of $\mathcal{S}$. Then there exists at least one fixpoint or at least one fixline.*

Proof. By our foregoing observations, either the number of points is odd, or the number of lines is odd. If the number of points is odd, than every involution fixes at least one point. If the number of lines is odd, than every involution fixes at least one line. □

If we have a thin octagon of order $(1,t)$, where $2t$ is compatible with the order of θ, then we obtain:

$$k_1(1+t+\sqrt{2t})+k_2(1+t-\sqrt{2t})+k_3(1+t)+2(1+t)=(1+t)f_0+f_1.$$

If $2t$ is not a square, then it follows that $k_1=k_2$, hence:

$$k_1 2(1+t)+k_3(1+t)+2(1+t)=(1+t)f_0+f_1.$$

Note that this implies that in a generalized quadrangle of order t, with $2t$ not a square, and such that $2t$ is compatible with the order of a duality, the number of absolute elements of that duality is divisible by $1+t$. But we will do better in Corollary 1.5.7.

Theorem 1.5.4 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that $2st$ is compatible with the order of θ (which is automatic when $\mathcal{S}$ is thick). If f_0, f_1 and f_2 are as before, then for the integers k_1, k_2 and k_3 obtained in* Theorem 1.5.1 *there holds*

$$k_1(s+t+\sqrt{2st})^2+k_2(s+t-\sqrt{2st})^2+k_3(s+t)^2+((1+s)(1+t))^2=$$
$$(1+s+t)(1+t)f_0+(1+s+2t)f_1+f_2.$$

Proof. This proof is totally analogous to the proof of Theorem 1.4.4. □

For a thin octagon of order $(1,t)$ such that, if $2t$ is not a perfect square, then $2t$ is compatible with the order of θ, we obtain:

$$k_1(1+t+\sqrt{2t})^2+k_2(1+t-\sqrt{2t})^2+k_3(1+t)^2+(2(1+t))^2=(2+t)(1+t)f_0+(2+2t)f_1+f_2.$$

If, in this case, $2t$ is not a square, then it follows that $k_1=k_2$ and we obtain:

$$k_1 2(1+4t+t^2)+k_3(1+t)^2+(2(1+t))^2=(2+t)(1+t)f_0+(2+2t)f_1+f_2.$$

Theorem 1.5.5 *Let $\mathcal{S}$ be a generalized octagon of order (s,t) and let θ be a nontrivial automorphism of $\mathcal{S}$. Suppose that $2st$ is compatible with the order of θ (which is automatic when $\mathcal{S}$ is thick). If f_0, f_1, f_2 and f_3 are defined as above, then for the integers k_1, k_2 and k_3 of* Theorem 1.5.1 *there holds*

$$k_1(s+t+\sqrt{2st})^3+k_2(s+t-\sqrt{2st})^3+k_3(s+t)^3+((1+s)(1+t))^3=$$
$$(s(s-1)(1+t)+3s(1+t)^2+(1+t)^3)f_0$$
$$+(s(1+t)+(s-1)^2+st+3(1+t)(s-1)+3(1+t)^2)f_1$$
$$+(2(s-1)+3(1+t))f_2+f_3.$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proofs of Theorems 1.4.1 and 1.4.4 we can prove that $\mathrm{tr}(QM^3) = k_1(s+t+\sqrt{2st})^3 + k_2(s+t-\sqrt{2st})^3 + k_3(s+t)^3 + ((1+s)(1+t))^3$, with k_1, k_2 and k_3 integers. On the other hand, because of Lemma 1.2.6 and the values for p^i_j given after that lemma, we can calculate that $A^3 = (a_{ij})$ is the matrix with $s(s-1)(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = s(1+t) + (s-1)^2 + st$ if $x_i \sim x_j$, $a_{ij} = 2(s-1)$ if $\mathrm{d}(x_i, x_j) = 2$, $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 3$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^3) = s(s-1)(1+t)f_0 + (s(1+t)+(s-1)^2+st)f_1 + 2(s-1)f_2 + f_3$. Because of the proof of Theorem 1.4.4 we know that $\mathrm{tr}(QA^2) = s(1+t)f_0 + (s-1)f_1 + f_2$, $\mathrm{tr}(QA) = f_1$ and $\mathrm{tr}(Q) = f_0$. Hence

$$
\begin{aligned}
& \mathrm{tr}(QM^3) \\
= \ & \mathrm{tr}(Q(A+(1+t)I)^3) \\
= \ & \mathrm{tr}(QA^3) + 3(1+t)\mathrm{tr}(QA^2) + 3(1+t)^2\mathrm{tr}(QA) + (1+t)^3\mathrm{tr}(Q) \\
= \ & s(s-1)(1+t)f_0 + (s(1+t)+(s-1)^2+st)f_1 + 2(s-1)f_2 + f_3 \\
& +3(1+t)(s(1+t)f_0 + (s-1)f_1 + f_2) + 3(1+t)^2 f_1 + (1+t)^3 f_0 \\
= \ & (s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0 \\
& +(s(1+t)+(s-1)^2+st+3(1+t)(s-1)+3(1+t)^2)f_1 \\
& +(2(s-1)+3(1+t))f_2 + f_3.
\end{aligned}
$$

Using Lemma 1.2.5 as before, the proof of the theorem is complete. $\square$

For a thin octagon with $s = 1$ and with $2t$ compatible with the order of θ, we obtain:

$$
\begin{gathered}
k_1(1+t+\sqrt{2t})^3 + k_2(1+t-\sqrt{2t})^3 + k_3(1+t)^3 + (2(1+t))^3 = \\
(3(1+t)^2 + (1+t)^3)f_0 + (1+2t+3(1+t)^2)f_1 + 3(1+t)f_2 + f_3.
\end{gathered}
$$

If $2t$ is not a square, then it follows that $k_1 = k_2$ and we obtain:

$$
\begin{gathered}
k_1 2(1+9t+9t^2+t^3) + k_3(1+t)^3 + (2(1+t))^3 = \\
(3(1+t)^2 + (1+t)^3)f_0 + (1+2t+3(1+t)^2)f_1 + 3(1+t)f_2 + f_3.
\end{gathered}
$$

Corollary 1.5.6 *Suppose that we have a thin octagon of order $(1,t)$. Consider a duality θ in the underlying generalized quadrangle. If $2t$ is not a square, and $2t$ is compatible with the order of θ, then $f_1 = 2(1+t)$. If $2t$ is a square, then $f_1 \equiv 2 \mod 2\sqrt{2}t$. In particular there is at least one absolute point and one absolute line.*

Proof. Since we have a duality in the underlying generalized quadrangle, we know that $f_0 = 0$ and $f_2 = 0$. Because of Lemma 1.2.5, Theorems 1.5.1, 1.5.4 and 1.5.5, we have the following equations:

$$\begin{cases} k_1(1+t+\sqrt{2t}) + k_2(1+t-\sqrt{2t}) + k_3(1+t) + 2(1+t) = f_1, \\ k_1(1+t+\sqrt{2t})^2 + k_2(1+t-\sqrt{2t})^2 + k_3(1+t)^2 + (2(1+t))^2 = (2+2t)f_1, \\ k_1(1+t+\sqrt{2t})^3 + k_2(1+t-\sqrt{2t})^3 + k_3(1+t)^3 + (2(1+t))^3 = \\ \qquad (1+2t+3(1+t)^2)f_1 + f_3. \end{cases}$$

Because f_0 and f_2 are 0, we know that $f_1 + f_3 = 2(1+t)(1+t^2)$. Hence:

$$\begin{cases} k_1 = \frac{\sqrt{2}(-2t+f_1-2)}{4\sqrt{t}}, \\ k_2 = -\frac{\sqrt{2}(-2t+f_1-2)}{4\sqrt{t}}, \\ k_3 = 0, \\ f_3 = 2(t^3+t^2+t+1) - f_1. \end{cases}$$

So $\frac{\sqrt{2}(-2t+f_1-2)}{4\sqrt{t}}$ has to be an integer. In the case that $2t$ is not a square, this only holds if $-2t + f_1 - 2 = 0$. Hence $f_1 = 2(1+t)$ if $2t$ is not a square. If $2t$ is a square, then $f_1 - 2$ has to be a multiple of $2\sqrt{2t}$. Hence $f_1 \equiv 2 \mod 2\sqrt{2t}$. □

From Lemma 1.2.7 now immediately follows.

Corollary 1.5.7 *Suppose that θ is a duality of a generalized quadrangle of order t. If $2t$ is not a square, and if $2t$ is compatible with the order of θ, then θ admits $1+t$ absolute points and $1+t$ absolute lines, and there are $(1+t)t^2$ points which are mapped to a line at distance 3 and $(1+t)t^2$ lines which are mapped to a point at distance 3. If $2t$ is a perfect square, then it has $1 \mod \sqrt{2t}$ absolute points and equally many absolute lines.*

If $2t$ is a square, then one can again construct examples of dualities in a generalized quadrangle of order t, namely in the symplectic quadrangle $\mathsf{W}(t)$, admitting the lower bound 1 of absolute points given in the previous corollary. Indeed, consider a polarity ρ and compose it with a nontrivial central root elation τ whose center is an absolute point x. The resulting duality $\theta = \rho\tau$ has x as unique absolute point. Indeed, suppose by way of contradiction that the point $y \neq x$ is absolute for θ. Since τ is involutive, this implies that $y^\rho \mathtt{I} y^\tau$. It is easy to see that x and y are not collinear and that $x \not\mathtt{I}\, y^\rho$. Since τ is central, the unique point z on y^ρ collinear with x is also collinear with y. Applying ρ to the chain $y \mathtt{I} yz \mathtt{I} z \mathtt{I} y^\rho$, we deduce that $z^\rho = yz$, hence z is an absolute point for ρ, contradicting the fact that z is collinear with x.

Remark 1.5.8 There are some restrictions on the parameter t of a self-dual generalized quadrangle of order t. Indeed, Theorem III.3 of [37] says that, if $t \equiv 2 \mod 8$, then no prime p dividing the square-free part of $2t$ may be congruent to $3 \mod 4$. This was improved by Haemers in [25] to: If $t \equiv 2 \mod 4$, then $2t$ must be a square. Also, if the generalized quadrangle is self-polar, than $2t$ is a square (see [36]).

1.6 Collineations of dodecagons and dualities of hexagons

Suppose that $\mathcal{S}$ is a generalized dodecagon. Let θ be an automorphism of $\mathcal{S}$ and let f_i be the number of points for which $\mathrm{d}(x, x^\theta) = i$ in the point graph, with $i \in \{0, 1, 2, 3, 4, 5\}$. The matrices M, A and Q are defined analogously as before. The eigenvalues of M are as follows (cf. [20]):

eigenvalues of M	multiplicity
0	m_0
$(s+1)(t+1)$	$m_1 = 1$
$s+t+\sqrt{st}$	m_2
$s+t-\sqrt{st}$	m_3
$s+t+\sqrt{3st}$	m_4
$s+t-\sqrt{3st}$	m_5
$s+t$	m_6

Thick finite generalized dodecagons do not exist, but nevertheless we formulate the following results with general s and t. In real life, either s or t is equal to 1, but the formulae do not seem to be equivalent. Afterwards, we apply our results to the case $s = 1$, implying results for dualities of generalized hexagons. In Chapter 6 we will also apply these formulae for $t = 1$.

Theorem 1.6.1 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that both st and $3st$ are compatible with the order of θ. If f_0 is the number of points fixed by θ and f_1 is the number of points x for which $x \sim x^\theta$, then for some integers k_1, k_2, k_3, k_4 and k_5 there holds*

$$k_1(s+t+\sqrt{st})+k_2(s+t-\sqrt{st})+k_3(s+t+\sqrt{3st})+k_4(s+t-\sqrt{3st})+k_5(s+t)+(1+s)(1+t) = (1+t)f_0+f_1.$$

Proof. The proof is totally analogous to the proof of Theorem 1.4.1. □

Note that we do need to check that both numbers st and $3st$ are compatible with the order of θ, even in the case where st is a prime power. Indeed, it is clear that $st=5$ is not compatible with 5, but $3st=15$ is compatible with 5. Also, $st=7$ is compatible with 21 (see Lemma 1.2.3), but, as $\sqrt{-3}$ belongs to the third cyclotomic extension of $\mathbb{Q}$ (see the proof of Lemma 1.2.3) and similarly $\sqrt{-7}$ belongs to the seventh cyclotomic extension of $\mathbb{Q}$, it follows that the product $\sqrt{21}$ belongs to the twenty-first cyclotomic extension of $\mathbb{Q}$ and hence $3st=21$ is not compatible with 21.

Theorem 1.6.2 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that both st and $3st$ are compatible with the order of θ. If f_i, $i=0,1,2$, is defined as above, then for the integers k_1, k_2, k_3, k_4 and k_5 obtained in* Theorem 1.6.1 *there holds*

$$k_1(s+t+\sqrt{st})^2+k_2(s+t-\sqrt{st})^2+k_3(s+t+\sqrt{3st})^2+k_4(s+t-\sqrt{3st})^2+k_5(s+t)^2+((1+s)(1+t))^2 = (1+s+t)(1+t)f_0+(1+s+2t)f_1+f_2.$$

Proof. This proof is totally analogous to the proof of Theorem 1.4.4. □

Theorem 1.6.3 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that both st and $3st$ are compatible with the order of θ. If f_i, $i=0,1,2,3$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 1.6.1 *there holds*

$$\begin{gathered} k_1(s+t+\sqrt{st})^3+k_2(s+t-\sqrt{st})^3+k_3(s+t+\sqrt{3st})^3+k_4(s+t-\sqrt{3st})^3+k_5(s+t)^3+ \\ ((1+s)(1+t))^3 = \\ (s(s-1)(1+t)+3s(1+t)^2+(1+t)^3)f_0 \\ +(s(1+t)+(s-1)^2+st+3(1+t)(s-1)+3(1+t)^2)f_1 \\ +(2(s-1)+3(1+t))f_2+f_3. \end{gathered}$$

Proof. This proof is totally analogous to the proof of Theorem 1.5.5. □

Theorem 1.6.4 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that both st and $3st$ are compatible with the order of θ. If f_i, $i = 0,1,2,3,4$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 1.6.1 *there holds*

$$\begin{gathered}
k_1(s+t+\sqrt{st})^4 + k_2(s+t-\sqrt{st})^4 + k_3(s+t+\sqrt{3st})^4 + k_4(s+t-\sqrt{3st})^4 + k_5(s+t)^4 + \\
((1+s)(1+t))^4 = \\
((s(1+t)+(s-1)^2+st)(1+t)s + 4s(s-1)(1+t)^2 + 6s(1+t)^3 + (1+t)^4)f_0 \\
+(s(s-1)(1+t) + (s-1)(s(1+t)+(s-1)^2+st) + 2(s-1)st + 4(1+t)(s(1+t) \\
+(s-1)^2+st) + 6(1+t)^2(s-1) + 4(1+t)^3)f_1 \\
+(s(1+t) + 3(s-1)^2 + 2st + 8(1+t)(s-1) + 6(1+t)^2)f_2 \\
+(3(s-1)+4(1+t))f_3 + f_4.
\end{gathered}$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proofs of Theorems 1.4.1, 1.4.4 and 1.5.5 we can prove that $\mathrm{tr}(QM^4) = k_1(s+t+\sqrt{st})^4 + k_2(s+t-\sqrt{st})^4 + k_3(s+t+\sqrt{3st})^4 + k_4(s+t-\sqrt{3st})^4 + k_5(s+t)^4 + ((1+s)(1+t))^4$, with k_1, k_2, k_3, k_4 and k_5 integers. On the other hand, because of Lemma 1.2.6 and the values for p^i_j given after that lemma, we can calculate that $A^4 = (a_{ij})$ is the matrix with $(s(1+t)+(s-1)^2+st)(1+t)s$ on the main diagonal while on the other entries we have

$$\begin{array}{ll}
a_{ij} = s(s-1)(1+t) + (s-1)(s(1+t)+(s-1)^2+st) + 2(s-1)st & \text{if } x_i \sim x_j \\
a_{ij} = s(1+t) + 3(s-1)^2 + 2st & \text{if } \mathrm{d}(x_i,x_j) = 2 \\
a_{ij} = 3(s-1) & \text{if } \mathrm{d}(x_i,x_j) = 3 \\
1 & \text{if } \mathrm{d}(x_i,x_j) = 4 \\
0 & \text{otherwise.}
\end{array}$$

Hence

$$\begin{aligned}
\mathrm{tr}(QA^4) = {}& ((s(1+t)+(s-1)^2+st)(1+t)s)f_0 \\
&+(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st)f_1 \\
&+(s(1+t)+3(s-1)^2+2st)f_2 + 3(s-1)f_3 + f_4.
\end{aligned}$$

The rest of the proof is totally analogous to the proof of Theorem 1.5.5. □

Theorem 1.6.5 *Let $\mathcal{S}$ be a generalized dodecagon of order (s,t) and let θ be an automorphism of $\mathcal{S}$. Suppose that both st and $3st$ are compatible with the order of θ. If f_i, $i=0,1,2,3,4,5$, is as before, then for the integers k_1, k_2, k_3, k_4 and k_5 of* Theorem 1.6.1 *there holds*

$$\begin{gathered}
k_1(s+t+\sqrt{st})^5+k_2(s+t-\sqrt{st})^5+k_3(s+t+\sqrt{3st})^5+k_4(s+t-\sqrt{3st})^5+k_5(s+t)^5+\\
((1+s)(1+t))^5=(s(t+1)(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st)+\\
5(1+t)(s(1+t)+(s-1)^2+st)(1+t)s+10(1+t)^2s(s-1)(1+t)+10(1+t)^3s(t+1)+(1+t)^5)f_0\\
+((s(1+t)+(s-1)^2+st)(1+t)s+(s-1)(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+\\
2(s-1)st)+st(s(1+t)+3(s-1)^2+2st)+5(1+t)(s(s-1)(1+t)+(s-1)(s(1+t)+\\
(s-1)^2+st)+2(s-1)st)+10(1+t)^2(s(1+t)+(s-1)^2+st)+10(1+t)^3(s-1)+\\
5(1+t)^4)f_1\\
+(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st+(s-1)(s(1+t)+3(s-1)^2+2st)\\
+3st(s-1)+5(1+t)(s(1+t)+3(s-1)^2+2st)+10(1+t)^2 2(s-1)+10(1+t)^3)f_2\\
+(s(1+t)+3(s-1)^2+2st+3(s-1)^2+st+5(t+1)3(s-1)+10(t+1)^2)f_3\\
+(4(s-1)+5(t+1))f_4+f_5.
\end{gathered}$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proofs of Theorems 1.4.1, 1.4.4 and 1.5.5 we can prove that $\mathrm{tr}(QM^5)=k_1(s+t+\sqrt{st})^5+k_2(s+t-\sqrt{st})^5+k_3(s+t+\sqrt{3st})^5+k_4(s+t-\sqrt{3st})^5+k_5(s+t)^5+((1+s)(1+t))^5$, with k_1, k_2, k_3, k_4 and k_5 the integers of Theorem 1.6.1 (by Lemma 1.2.5). On the other hand, because of Lemma 1.2.6 and the values for p^i_j given after that lemma, we can calculate that $A^4=(a_{ij})$ is the matrix with $s(t+1)(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st)$ on the main diagonal while on the other entries we have

$$\begin{array}{ll}
a_{ij}=(s(1+t)+(s-1)^2+st)(1+t)s+(s-1)(s(s-1)(1+t) & \\
\qquad +(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st) & \\
\qquad +st(s(1+t)+3(s-1)^2+2st) & \text{if } x_i\sim x_j\\
a_{ij}=s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st & \\
\qquad +(s-1)(s(1+t)+3(s-1)^2+2st)+3st(s-1) & \text{if } \mathrm{d}(x_i,x_j)=2\\
a_{ij}=s(1+t)+3(s-1)^2+2st+3(s-1)^2+st & \text{if } \mathrm{d}(x_i,x_j)=3\\
a_{ij}=4(s-1) & \text{if } \mathrm{d}(x_i,x_j)=4\\
1 & \text{if } \mathrm{d}(x_i,x_j)=5\\
0 & \text{otherwise.}
\end{array}$$

Hence

$$\begin{aligned}
\mathrm{tr}(QA^5) = {} & (s(t+1)(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st))f_0 \\
& +((s(1+t)+(s-1)^2+st)(1+t)s+(s-1)(s(s-1)(1+t)+(s-1)(s(1+t) \\
& +(s-1)^2+st)+2(s-1)st)+st(s(1+t)+3(s-1)^2+2st))f_1 \\
& +(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st \\
& +(s-1)(s(1+t)+3(s-1)^2+2st)+3st(s-1))f_2 \\
& +(s(1+t)+3(s-1)^2+2st+3(s-1)^2+st)f_3+4(s-1)f_4+f_5.
\end{aligned}$$

We have

$$\begin{aligned}
& \mathrm{tr}(QM^5) \\
= {} & \mathrm{tr}(Q(A+(1+t)I)^5) \\
= {} & \mathrm{tr}(QA^5)+5(1+t)\mathrm{tr}(QA^4)+10(1+t)^2\mathrm{tr}(QA^3)+10(1+t)^3\mathrm{tr}(QA^2) \\
& +5(1+t)^4\mathrm{tr}(QA)+(1+t)^5\mathrm{tr}(Q).
\end{aligned}$$

If we substitute the formula for $\mathrm{tr}(QA^5)$ which we obtained above and the formulas which we obtained in Theorems 1.4.1, 1.4.4, 1.5.5 and 1.6.4 for $\mathrm{tr}(QA^4)$, $\mathrm{tr}(QA^3)$, $\mathrm{tr}(QA^2)$, $\mathrm{tr}(QA)$ and $\mathrm{tr}(Q)$, then we obtain the assertion. □

Corollary 1.6.6 *Suppose that we have a thin dodecagon of order $(1,t)$. Consider a duality θ in the underlying generalized hexagon. Suppose that both t and $3t$ are compatible with the order of θ. If $3t$ and t are no squares, then $f_1 = 2(1+t)$, $f_3 = 2(t^2+t^3)$ and $f_5 = 2(t^4+t^5)$. If $3t$ is a square, then $f_1 \equiv 2 \mod 2\sqrt{3t}$ and $f_3 \equiv 0 \mod 2\sqrt{3t}$. If t is a square, then $f_1 \equiv 2 \mod 2\sqrt{t}$ and $f_3 \equiv 0 \mod 2\sqrt{t}$. In particular there is always at least one absolute point and one absolute line.*

Proof. Since we have a duality in the underlying generalized hexagon, we know that $f_0 = 0$, $f_2 = 0$ and $f_4 = 0$. Because of Theorems 1.6.1, 1.6.2, 1.6.3, 1.6.4 and 1.6.5, we have the following equations:

$$\begin{cases}
k_1(1+t+\sqrt{t})+k_2(1+t-\sqrt{t})+k_3(1+t+\sqrt{3t})+k_4(1+t-\sqrt{3t})+k_5(1+t) \\
\qquad +2(1+t)=f_1, \\
k_1(1+t+\sqrt{t})^2+k_2(1+t-\sqrt{t})^2+k_3(1+t+\sqrt{3t})^2+k_4(1+t-\sqrt{3t})^2+k_5(1+t)^2 \\
\qquad +(2(1+t))^2=(2+2t)f_1, \\
k_1(1+t+\sqrt{t})^3+k_2(1+t-\sqrt{t})^3+k_3(1+t+\sqrt{3t})^3+k_4(1+t-\sqrt{3t})^3+k_5(1+t)^3 \\
\qquad +(2(1+t))^3=(1+2t+3(1+t)^2)f_1+f_3, \\
k_1(1+t+\sqrt{t})^4+k_2(1+t-\sqrt{t})^4+k_3(1+t+\sqrt{3t})^4+k_4(1+t-\sqrt{3t})^4+k_5(1+t)^4 \\
\qquad +(2(1+t))^4=(4(1+t)(1+2t)+4(1+t)^3)f_1+4(1+t)f_3, \\
k_1(1+t+\sqrt{t})^5+k_2(1+t-\sqrt{t})^5+k_3(1+t+\sqrt{3t})^5+k_4(1+t-\sqrt{3t})^5+k_5(t+1)^5 \\
\qquad +(2(t+1))^5=(5t^4+40t^3+85t^2+64t+16)f_1+(10t^2+24t+11)f_3+f_5.
\end{cases}$$

Because f_0, f_2 and f_4 are 0, we know that $f_1+f_3+f_5=\frac{2(t^6-1)}{t-1}$. Hence:

$$\begin{cases} k_1 = -\frac{-2t^3-f_1t+f_3+f_1-2}{4\sqrt{t^3}}, \\ k_2 = \frac{-2t^3-f_1t+f_3+f_1-2}{4\sqrt{t^3}}, \\ k_3 = \frac{(-2t^3-4t^2-4t-2+f_1t+f_3+f_1)\sqrt{3}}{12\sqrt{t^3}}, \\ k_4 = -\frac{(-2t^3-4t^2-4t-2+f_1t+f_3+f_1)\sqrt{3}}{12\sqrt{t^3}}, \\ k_5 = 0, \\ f_5 = 2(t^5+t^4+t^3+t^2+t+1)-f_1-f_3. \end{cases}$$

So $\frac{-2t^3-f_1t+f_3+f_1-2}{4\sqrt{t^3}}$ and $\frac{(-2t^3-4t^2+f_1t-2-4t+f_3+f_1)\sqrt{3}}{12\sqrt{t^3}}$ have to be integers. In the case that $3t$ and t are no squares, this only holds if $-2t^3-f_1t+f_3+f_1-2=0$ and $-2t^3-4t^2-4t-2+f_1t+f_3+f_1=0$. Hence $f_1=2(1+t)$ and $f_3=2(t^2+t^3)$ if $3t$ and t are no squares. If $3t$ is a square (so t is no square), then $-2t^3-f_1t+f_3+f_1-2=0$ and $-2t^3-4t^2+f_1t-2-4t+f_3+f_1$ has to be a multiple of $4t\sqrt{3t}$. Combining these, we see that f_1-2 has to be a multiple of $2\sqrt{3t}$, which means that $f_1 \equiv 2 \mod 2\sqrt{3t}$. Substituting this in the former equality yields $f_3 \equiv 0 \mod 2\sqrt{3t}$. On the other hand, if t is a square (so $3t$ is no square), then $-2t^3-4t^2+f_1t-2-4t+f_3+f_1=0$ and $-2t^3-f_1t+f_3+f_1-2$ has to be a multiple of $4t\sqrt{t}$. Hence f_1-2 has to be a multiple of $2\sqrt{t}$, which means that $f_1 \equiv 2 \mod 2\sqrt{t}$. Similarly as above, $f_3 \equiv 0 \mod 2\sqrt{t}$. □

This immediately implies, in view of Lemma 1.2.7, the following corollary.

Corollary 1.6.7 *Suppose that θ is a duality of a generalized hexagon of order t and suppose that both t and $3t$ are compatible with the order of θ. If none of $3t$ and t are perfect squares, then θ has $1+t$ absolute points and $1+t$ absolute lines, there are t^2+t^3 points which are mapped to a line at distance 3 and t^2+t^3 lines which are mapped to a point at distance 3, and there are t^4+t^5 points which are mapped to a line at distance 5 and t^4+t^5 lines which are mapped to a point at distance 5. If t is a perfect square, then there are $1 \mod \sqrt{t}$ absolute points and equally many absolute lines; the number of points mapped onto a line at distance 3 in the incidence graph is divisible by $\sqrt{t}$. If $3t$ is a perfect square, then there are $1 \mod \sqrt{3t}$ absolute points and equally many absolute lines; the number of points mapped onto a line at distance 3 in the incidence graph is divisible by $\sqrt{3t}$.*

We currently do not know of any finite self-dual generalized hexagon of order t, with neither t nor $3t$ a perfect square. If $3t$ is a square, then similarly as for symplectic quadrangles of order s, with $2s$ a square, one can easily construct dualities of the split

Cayley hexagon $\mathsf{H}(t)$ with exactly one absolute point as the composition of a polarity with a nontrivial central collineation with center one of the absolute points of the polarity. Note that, in this example, the order of the duality is equal to 6, and $3t$ is always compatible with 6.

Remark 1.6.8 Haemers proved in [25] that a self-dual generalized hexagon of order t does not exist if $t \equiv 2 \mod 4$. Also, Ott proved in [35] that a self-polar generalized hexagon of order t only exists if $3t$ is a square.

2

Symmetric designs and near hexagons

In this chapter we will prove a Benson-type theorem for designs and also for near hexagons of order $(1, t; \lambda, 1)$. As we will observe, these near hexagons are the doubles of the symmetric $2-(v, t+1, \lambda+1)$-designs. It will turn out that the formula obtained for the designs does not give anything new, but the formula for the near-hexagons applied to dualities of a symmetric 2-design does give us new information about the number of absolute points. This will be illustrated with a few examples. Note that these examples also include collineations of particular geometries, which can be considered as dualities of a suitably defined 2-design. However, in the cases where we start with a generalized quadrangle or a generalized hexagon, all the restrictions that we obtain can be deduced from the ones we already obtained in Chapter 1. New results are obtained for dualities of projective spaces, and for collineations of polar spaces with as many points on a line as there are maximal subspaces through a next-to-maximal subspace. In the case of rank at least 3, these are precisely the polar spaces arising from parabolic quadrics and symplectic polarities.

2.1 A Benson-type theorem for 2-designs

The following notation is analogous to the notation of Chapter 1. Suppose that $\mathcal{D}$ is a $2-(v,t+1,\lambda+1)$-design (for the notation related to designs see section 0.6; we will also use the standard notation r for the number of blocks through a point). Let D be an incidence matrix of $\mathcal{D}$. Then $M := DD^T = A+bI$, where A is an adjacency matrix of the point graph of $\mathcal{D}$ (note that this point graph admits multiple edges, see Subsection 0.6.1). Let θ be an automorphism of $\mathcal{D}$ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise; so Q is a permutation matrix. Since $M = A+rI$, and since this is clearly equal to $(r-\lambda-1)I + (\lambda+1)J$, where J is the all-one-matrix of the appropriate dimension, we see that the eigenvalues of M are as follows:

eigenvalues of M	multiplicity
$r+(v-1)(\lambda+1)$	$m_0 = 1$
$r-\lambda-1$	$m_1 = v-1$

Theorem 2.1.1 *Let $\mathcal{D}$ be a $2-(v,t+1,\lambda+1)$-design and let θ be an automorphism of $\mathcal{D}$. If f_0 is the number of points fixed by θ and if f_1 is the number of points x for which $x^\theta \neq x$, then for some integer k_0 there holds*

$$\mathrm{tr}(QM) = k_0(r-\lambda-1) + r + (v-1)(\lambda+1) = rf_0 + (\lambda+1)f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^nM^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (r+(v-1)(\lambda+1))J$, we have $(QM)J = (r+(v-1)(\lambda+1))J$, so $r+(v-1)(\lambda-1)$ is an eigenvalue of QM. Because $m_0 = 1$, it follows that this eigenvalue of QM has multiplicity 1. For each divisor d of n, let ξ_d denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1,2,\ldots,d-1\}$ that are relatively prime to d. Then U_d is an integer by [31]. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = r-\lambda-1$, because of Lemma 1.2.1. Let a_d denote the multiplicity of $\xi_d(r-\lambda-1)$ as an eigenvalue of QM, with $d|n$, and ξ_d a primitive d^{th} root of unity. Then we have:

$$\operatorname{tr}(QM) = \sum_{d|n} a_d(r-\lambda-1)U_d + r + (v-1)(\lambda+1),$$

or

$$\operatorname{tr}(QM) = k_0(r-\lambda-1) + r + (v-1)(\lambda+1),$$

with k_0 an integer.
Since the entry on the i^{th} row and i^{th} column of QM is the number of blocks incident with x_i and x_i^{θ}, we have $\operatorname{tr}(QM) = rf_0 + (\lambda+1)f_1$. Hence

$$k_0(r-\lambda-1) + r + (v-1)(\lambda+1) = rf_0 + (\lambda+1)f_1,$$

with k an integer. □

At first sight, this theorem might give additional information about collineations of 2-designs, but a closer look reveals that, substituting $f_1 = v - f_0$, we necessarily have $k_0 = f_0 - 1$. Hence we do not obtain anything new. However, if we look for a Benson-type formula for dualities, we will find new restrictions. That is what we do in the next section.

2.2 Dualities of symmetric designs

We recall that the double of a symmetric $2-(v, t+1, \lambda+1)$-design is a near hexagon of order $(1, t; \lambda+1)$ see Section 0.6.2. Recall also that now $r = t+1$.

If the matrix M of this near hexagon is defined as before, then it has the following eigenvalues:

eigenvalues of M	multiplicity
$2t+2$	$m_0 = 1$
0	$m_1 = 1$
$t+1+\sqrt{t-\lambda}$	$m_2 = v-1 = \frac{(t+1)t}{\lambda+1}$
$t+1-\sqrt{t-\lambda}$	$m_3 = v-1 = \frac{(t+1)t}{\lambda+1}$

Theorem 2.2.1 *Let $\mathcal{S}$ be a near hexagon of order $(1, t; \lambda + 1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that $t - \lambda$ is compatible with the order of θ. If g_0 is the number of points fixed by θ and g_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then for some integers k_1 and k_2 holds*

$$k_1(1 + t + \sqrt{t - \lambda}) + k_2(1 + t - \sqrt{t - \lambda}) + 2(1 + t) = (1 + t)g_0 + g_1.$$

Proof. This proof is completely similar to the proof of Theorem 1.3.1. □

Theorem 2.2.2 *Let $\mathcal{S}$ be a near hexagon of order $(1, t; \lambda + 1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that $t - \lambda$ is compatible with the order of θ. If g_0 is the number of points fixed by θ, g_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$ and g_2 is the number of points for which $\mathrm{d}(x, x^\theta) = 4$, then for some integers k_1 and k_2 holds*

$$k_1(1 + t + \sqrt{t - \lambda})^2 + k_2(1 + t - \sqrt{t - \lambda})^2 + (2(1 + t))^2 = (2 + t)(1 + t)g_0 + (2 + 2t)g_1 + g_2.$$

Proof. Suppose that M, A and Q are defined as before. Suppose that θ has order n, so that $(QM^2)^n = Q^n M^{2n} = M^{2n}$. It follows that the eigenvalues of QM^2 are the eigenvalues of M^2 multiplied by the appropriate roots of unity. Since $M^2 J = (2(1 + t))^2 J$, we have $(QM^2)J = (2(1 + t))^2 J$, so $(2(1 + t))^2$ is an eigenvalue of QM^2. By Lemma 1.2.5 $m_0 = 1$ and it follows that this eigenvalue of QM^2 has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM^2 with multiplicity $m_1 = 1$.

Using Lemma 1.2.5 and arguing as in the proof of Theorem 1.4.4, we obtain

$$\mathrm{tr}(QM^2) = k_1(1 + t + \sqrt{t - \lambda})^2 + k_2(1 + t - \sqrt{t - \lambda})^2 + (2(1 + t))^2,$$

with k_1 and k_2 the same integers appearing in the statement of Theorem 2.2.1. On the other hand we have

$$\begin{array}{rl} & M = A + (1 + t)I \\ \Rightarrow & QM = QA + (1 + t)Q \\ \Rightarrow & \mathrm{tr}(QM) = \mathrm{tr}(QA) + (1 + t)\mathrm{tr}(Q) \\ \Rightarrow & (1 + t)g_0 + g_1 = \mathrm{tr}(QA) + (1 + t)g_0 \\ \Rightarrow & \mathrm{tr}(QA) = g_1. \end{array}$$

The matrix $A^2 = (a_{ij})$ is the matrix with $(1 + t)$ along the main diagonal and on the other entries we have $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 4$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^2) = (1 + t)g_0 + g_2$. It follows that

$$\begin{aligned}
& \mathrm{tr}(QM^2) \\
= \ & \mathrm{tr}(Q(A+(1+t)I)^2) \\
= \ & \mathrm{tr}(QA^2) + 2(1+t)\mathrm{tr}(QA) + (1+t)^2\mathrm{tr}(Q) \\
= \ & (1+t)g_0 + g_2 + 2(1+t)g_1 + (1+t)^2 g_0 \\
= \ & (2+t)(1+t)g_0 + 2(1+t)g_1 + g_2.
\end{aligned}$$

□

We emphasize once again that the integers k_1 and k_2 in Theorems 2.2.1 and 2.2.2 are the same by Lemma 1.2.5.

Suppose that, under the same assumptions of the previous theorems, θ is a duality in the underlying symmetric design, then we know that $g_0 = 0$ and $g_2 = 0$. Because of Theorems 2.2.1 and 2.2.2, we have the following equations:

$$\begin{cases} k_1(1+t+\sqrt{t-\lambda}) + k_2(1+t-\sqrt{t-\lambda}) + 2(1+t) = g_1, \\ k_1(1+t+\sqrt{t-\lambda})^2 + k_2(1+t-\sqrt{t-\lambda})^2 + (2(1+t))^2 = (2+2t)g_1. \end{cases}$$

Hence:

$$k_1 = -k_2 = \tfrac{2t+2-g_1}{2\sqrt{t-\lambda}}.$$

So $\frac{2t+2-g_1}{2\sqrt{t-\lambda}}$ has to be an integer. In the case that $t-\lambda$ is not a square, this only holds if $g_1 = 2(t+1)$. Suppose that $t-\lambda$ is a square, then $g_1 - 2(t+1)$ should be a multiple of $2\sqrt{t-\lambda}$. Hence

$$\begin{aligned}
g_1 & \equiv 2(t+1) \mod 2\sqrt{t-\lambda} \\
& \equiv 2(1+\lambda) \mod 2\sqrt{t-\lambda}.
\end{aligned}$$

Corollary 2.2.3 *Suppose that θ is a duality of a symmetric $2-(v, t+1, \lambda+1)$-design. If $t-\lambda$ is not a square and $t-\lambda$ is compatible with the order of θ, then θ has $1+t$ absolute points and $1+t$ absolute lines. If $t-\lambda$ is a square, then it has $1+t \mod \sqrt{t-\lambda}$, or, equivalently, $1+\lambda \mod \sqrt{t-\lambda}$ absolute points and equally many absolute lines.*

2.3 Benson-type formulas for designs applied to some examples

2.3.1 The symmetric design $\mathsf{PG}_{n-1}(n,q)$

Consider the symmetric design $\mathsf{PG}_{n-1}(n,q)$; the point set of this design is the point set of $\mathsf{PG}(n,q)$ and the block set is the set of hyperplanes of $\mathsf{PG}(n,q)$, with natural incidence relation. In our notation this is a $2-(\frac{q^{n+1}-1}{q-1}, \frac{q^n-1}{q-1}, \frac{q^{n-1}-1}{q-1})$-design. Suppose that we have a duality θ in this design. We know that $t-\lambda = q^{n-1}$ is no square if q is an odd power of a prime p and n is even. In this case, and if p is compatible with the order of θ, we necessarily have $g_1 = 2t+2$, which means that there are $t+1 = \frac{q^n-1}{q-1}$ absolute points and $t+1 = \frac{q^n-1}{q-1}$ absolute hyperplanes.

An example of such a duality is an orthogonal polarity (q odd) or a pseudo polarity (q even). The above result "explains" why the number of points of a parabolic quadric is equal to the number of points of a hyperplane. But it says more. Indeed, *almost every* duality in this projective space must have the same number of absolute points. Given the fact that every duality is associated to a nonsingular matrix $T = (t_{ij})_{0\leq i\leq n, 0\leq j\leq n}$ and a field automorphism σ, this immediately implies the following algebraic corollary.

Corollary 2.3.1 *Let q be a non-square power of a prime p and let n be even. Let $T = (t_{ij})_{0\leq i\leq n, 0\leq j\leq n}$ be a nonsingular $(n+1)\times(n+1)$-matrix and let σ be an automorphism of $\mathsf{GF}(q)$. Suppose that p is compatible with the order of the semi-linear mapping determined by T and the companion field automorphism σ. Then the equation*

$$\sum_{i,j=0}^{n} t_{ij} x_i x_j^{\sigma} = 0$$

in the unknowns $x_0, x_1, \ldots, x_n$ has exactly q^n solutions over $\mathsf{GF}(q)$.

The number q^n comes from $\frac{q^n-1}{q-1}$ absolute points; each point gives rise to $q-1$ solutions (using scalar multiples of the coordinates). Then also add the zero-solution.

2.3.2 Parabolic quadrics and symplectic polar spaces

Let $\mathfrak{P}$ be either a parabolic quadric in the projective space $\mathsf{PG}(2n,q)$ or a symplectic polar space in a projective space $\mathsf{PG}(2n-1,q)$, $n \geq 2$. Let $\mathcal{D}$ be the design defined as follows. The points of $\mathcal{D}$ are the points of $\mathfrak{P}$; the blocks of $\mathcal{D}$ are also the points of $\mathcal{D}$ (but thought of as belonging to a "second copy"); incidence is collinearity (where equality is included). An elementary count reveals that this structure is indeed a symmetric 2-design with $v = \frac{q^{2n}-1}{q-1}$ points and equally many blocks, each block has $t+1 = \frac{q^{2n-1}-1}{q-1}$ points and each point is contained in equally many blocks, two blocks intersect in exactly $\lambda + 1 = \frac{q^{2n-2}-1}{q-1}$ points and two points lie in equally many common blocks.

Then $t - \lambda = q^{2n-2}$ is a perfect square. Let θ be a collineation of $\mathfrak{P}$, as a polar space. Then θ induces a uniquely defined duality of the associated design $\mathcal{D}$ in the obvious way. We denote this duality by θ^*. The definition of $\mathcal{D}$ readily implies that a point x of $\mathfrak{P}$ is mapped onto a collinear point (including the possibility of being fixed) by θ if and only if x, as a point of $\mathcal{D}$, is an absolute point for θ^*. According to Corollary 2.2.3, the number g_1 of absolute elements of θ^* satisfies $g_1 \equiv 2(q^{n-2} + q^{n-3} + \cdots + 1) \mod 2q^{n-1}$. Since there are equally many absolute points as absolute lines, we conclude the following.

Corollary 2.3.2 *Let θ be a collineation of either a parabolic quadric in the projective space $\mathsf{PG}(2n, q)$ or a symplectic polar space in a projective space $\mathsf{PG}(2n-1, q)$, $n \geq 2$. Then θ maps $q^{n-2} + q^{n-3} + \cdots + 1 \mod q^{n-1}$ points to a collinear or equal point, and hence $0 \mod q^{n-1}$ points are mapped onto an opposite one.*

So, on the one hand, the previous corollary does not exclude the possibility for a collineation of a polar space to map no point to an opposite. In Chapter 8 we will characterize such collineations and give examples. On the other hand, it also says that every collineation of the (finite) polar spaces in question must map at least one point to a non-opposite one. One could wonder whether this is a general fact. There are certainly counterexamples to this statement in the infinite case, even for parabolic quadrics. Indeed, just consider a parabolic quadric in $\mathsf{PG}(2n, \mathbb{C})$ given by an equation with real coefficients which is anisotropic over $\mathbb{R}$. Then complex conjugation can not map a point onto a collinear one as otherwise the joining line must be a real isotropic one. In the finite case, one can show the following assertion:

Proposition 2.3.3 *Every collineation of any finite polar space of rank at least 3 maps at least one point to a non-opposite one.*

Proof. Suppose, by way of contradiction, that the collineation θ of some finite polar space of rank at least 3 maps every point to an opposite one. Consider a plane π of the polar space. Then its image under θ must be an opposite plane, as otherwise some point x of

π is collinear to all points of π^θ, and hence also to its image! The map that sends a point or line of π to the projection onto π of its image under θ is now a duality of π without absolute points. This contradicts Corollary 1.4.6 of Chapter 1. □

There remains to consider finite generalized quadrangles, say of order (s, t). But in this case we have Corollary 1.3.3 that tells us that, if s and t are not relatively prime, then every collineation has a point mapped onto a non-opposite point. If t and s are relatively prime, then there are obvious counterexamples. Indeed, consider a generalized quadrangle of type $T^*(O)$, with O a hyperoval of some plane $\mathsf{PG}(2,q)$, considered as plane at infinity of some affine space $\mathsf{AG}(3,q)$. The points of the quadrangle are the points of the affine space $\mathsf{AG}(3,q)$, and the lines are the lines of $\mathsf{AG}(3,q)$ meeting $\mathsf{PG}(2,q)$ in a point of the hyperoval O. Then the collineation induced by a translation in $\mathsf{AG}(3,q)$ with center a point off O does not map some point to a non-opposite one. Note that the order of the quadrangle is $(q-1, q+1)$ and that $q-1$ and $q+1$ are indeed relatively prime, as q is necessarily even.

This completely solves the question of existence of collineations of polar spaces mapping no point to a non-opposite one.

2.3.3 The symmetric design which arises from a generalized quadrangle of order (s, s)

Suppose that we have a generalized quadrangle $\mathcal{S}$ of order (s, s). We obtain a design from this generalized quadrangle in the following way. The points of the design are the points of the generalized quadrangle, the blocks of the design are also the points of the generalized quadrangle and the incidence relation is collinearity in the generalized quadrangle (note that in this case equality is again included). Hence we obtain a $2-(s^3+s^2+s+1, s^2+s+1, s+1)$-design. The double of this design is a near hexagon of order $(1, s^2+s; s+1)$. We will now apply Corollary 2.2.3. Because s^2 is a square, every duality of this design has $1 \mod s$ absolute points and $1 \mod s$ absolute lines. Hence, just like in the previous subsection, this implies that every collineation of $\mathcal{S}$ maps $1 \mod s$ points to a non-opposite point.

When we consider Benson's original theorem applied to generalized quadrangles of order (s, s), we obtain the following. Let θ be a collineation of $\mathcal{S}$, and let f_0 and f_1 be the number of points fixed under θ and the number of points mapped onto a collinear point, respectively; then $(1+s)f_0 + f_1 \equiv 1 + s^2 \mod 2s$. Taking this formula modulo s, we obtain the above result. Hence, Benson's original theorem is slightly more precise than the derivative for symmetric 2-designs and near-hexagons applied to generalized quadrangles.

2.3.4 The symmetric design which arises from a generalized quadrangle of order $(q+1, q-1)$

Suppose that we have a generalized quadrangle $\mathcal{S}$ of order $(q+1, q-1)$. We obtain a design from this generalized quadrangle in the following way. The points of the design are the points of the generalized quadrangle, the blocks of the design are also the points of the generalized quadrangle and the incidence relation is collinearity in the generalized quadrangle, but this time equality is not included. Hence we obtain a $2-(q^2(q+2), q^2+q, q)$-design. The double of this design is a near hexagon of order $(1, q^2+q-1; q)$. When we apply Corollary 2.2.3 we obtain, because q^2 is a square, that every duality of this design has $0 \mod q$ absolute points and $0 \mod q$ absolute lines. Hence every collineation of $\mathcal{S}$ maps $0 \mod q$ points to a collinear, but distinct point. If f_0 is the number of fixed points, and f_1 is the number of points mapped onto a collinear, but distinct point, then Benson's original theorem states that $qf_0 + f_1 \equiv q^2 \mod 2q$. This provides again slightly more information and we reach the same conclusion as in the previous paragraph.

2.3.5 The symmetric design which arises from a generalized hexagon of order (s, s)

Suppose that we have a generalized hexagon $\mathcal{H}$ of order (s, s). We construct a symmetric design in the following way. The points of the design are the points of the generalized hexagon, the block set of the design is a second copy of the points set of the generalized hexagon and incidence is being not opposite in the generalized hexagon. Now, the number of points in a hexagon of order s not opposite two collinear points is a constant only depending on s (and not on the hexagon), and also the number of points not opposite two given non-collinear non-opposite points is a constant depending only on s (and these dependencies are given by polynomials in s). But these constants must be the same because of the fact that the split Cayley hexagon of order q has the same point set as a parabolic quadric of rank 3, and because opposite points of the split Cayley hexagon correspond to non-collinear points of the parabolic quadric, and so in this case the design is the same as the one defined in Subsection 2.3.2 above for $n = 3$. Hence we obtain a $2-(1+s+s^2+s^3+s^4+s^5, 1+s+s^2+s^3+s^4, 1+s+s^2+s^3)$-design $\mathcal{D}$.

So, with the standard notation, $t - \lambda = s^4$ is a perfect square. Let θ be a collineation of $\mathcal{H}$, as a generalized hexagon. Then θ induces a uniquely defined duality of the associated design $\mathcal{D}$ in the obvious way. We denote this duality by θ^*. The definition of $\mathcal{D}$ readily implies that a point x of $\mathcal{H}$ is mapped onto a non-opposite point by θ if and only if x, as a point of $\mathcal{D}$, is an absolute point for θ^*. According to Corollary 2.2.3, the number g_1 of absolute elements of θ^* satisfies $g_1 \equiv 2(s+1) \mod 2s^2$. Since there are equally many absolute points as absolute lines, we conclude the following.

Corollary 2.3.4 *Let θ be a collineation of a generalized hexagon of order s. Then θ maps $s+1 \mod s^2$ points to a non-opposite point, and hence $0 \mod s^2$ points are mapped onto an opposite one.*

Clearly, this corollary does not follow from Theorem 1.4.4. But it does follow from that theorem combined with Theorem 1.4.1. Indeed, by putting $s = t$ in the expression in Theorem 1.4.4 and by considering it modulo s^2, we see that we can write the obtained congruence as

$$(f_0 + f_1 + f_2) + 3s(f_0 + f_1) \equiv (1 + s) + 3s \mod s^2.$$

Since by Theorem 1.4.1, $f_0 + f_1 \equiv 1 \mod s$, so $3s(f_0 + f_1) \equiv 3s \mod s^2$, we obtain the above corollary.

Still, the corollary puts forward the question of whether any collineation of any generalized hexagon can map every point to an opposite. For hexagons of order s, this is impossible, just by the corollary. For infinite hexagons, there are again counterexamples, much in the same spirit as the ones in Subsection 2.3.2, using the complex split Cayley hexagon and the parabolic quadric $\mathsf{PG}(6, \mathbb{C})$. In the general finite case, the results of Theorem 1.4.1 and Theorem 1.4.4 do not give a conclusive answer: again, if (s, t) is the order, then s, t being not relatively prime implies the nonexistence of the wanted collineation. But if s and t are relatively prime, we cannot turn to counterexamples, as there are no hexagons known with such parameters. Besides the usual divisibility conditions given by the multiplicity of the eigenvalues (see page 43 in Chapter 1), we also obtain in this case that $2(s+t+\sqrt{st})$ must divide $(1+s)(1+t)(1+\sqrt{st})$ (if such a collineation would exist). Indeed, Theorems 1.4.1 and 1.4.4 imply (noting that $f_0 = f_1 = f_2 = 0$) that the integers k_1 and k_2 are determined by the following system of equations:

$$\begin{cases} k_1(s + t + \sqrt{st}) + k_2(s + t - \sqrt{st} & = & -(1 + s)(1 + t), \\ k_1(s + t + \sqrt{st})^2 + k_2(s + t - \sqrt{st}^2 & = & -(1 + s)^2(1 + t)^2. \end{cases}$$

Solving for k_1 gives us the desired divisibility condition.

3

Partial geometries and near octagons

In this chapter, we first prove some general results on the number of fixed points of collineations of finite partial geometries, and on the number of absolute points of dualities of partial geometries. The only known candidates of self-dual partial geometries, in other words, the only known candidates of partial geometries admitting dualities, or, still in other words, the only known proper partial geometries with parameters (t, t, α), are a partial geometry with parameters $(5, 5, 2)$ or arise from maximal arcs by constructions of Thas [47] and Mathon [33]. An interesting class of examples are the partial geometries arising from a Thas 1974 maximal arc of a Desarguesian projective plane constructed with a Suzuki-Tits ovoid. In the second part of the present chapter, we show (1) that these examples are really self-dual (in fact we show that this holds when considering any ovoid of $\mathsf{PG}(3, q)$, with q even). Our methods then allow to (2) determine the full collineation groups of these geometries. As an application we show (3) that, for each Suzuki-Tits ovoid, there are exactly two isomorphism classes of Thas 1974 maximal arcs in the classical plane, and consequently also two isomorphism classes of corresponding partial geometries. Question (3) was also answered by Hamilton & Penttila [27], see also a note in [26]. We include a proof not only for completeness' sake, but also since we believe that the authors of [27] tacitly assumed something which requires an explicit proof (see below for details). Question (1) was, as far as we know, never treated before and open since 1974, when Thas introduced these geometries.

3.1 A Benson-type theorem for partial geometries

Let A be an adjacency matrix of the point graph of a partial geometry $\mathcal{S}$ of order (s, t, α) with v points, let $M = A + (t+1)I$, let θ be an automorphism of $\mathcal{S}$ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise. Recall that $QM = MQ$; see Section 1.2.

We will use the notation and results introduced and proved in Chapter 1 Section 1.2. In particular we draw the reader's attention to Lemmas 1.2.1 and 1.2.5.

We now introduce some further notation. Suppose that $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a partial geometry of order (s, t, α). It is convenient to use the notion of collinearity only for distinct points. Let D be an incidence matrix of $\mathcal{S}$. Then $M := DD^T = A + (t+1)I$, where A is an adjacency matrix of the point graph of $\mathcal{S}$. Let θ be an automorphism of $\mathcal{S}$ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise; so Q is a permutation matrix. Because $M = A + (t+1)I$, the eigenvalues of M are as follows (cfr. [6]):

eigenvalues of M	multiplicity
0	$m_0 = \frac{s(s+1-\alpha)(st+\alpha)}{\alpha(s+t+1-\alpha)}$
$(s+1)(t+1)$	$m_1 = 1$
$s+t+1-\alpha$	$m_2 = \frac{(s+1)(t+1)st}{\alpha(s+t+1-\alpha)}$

Since all eigenvalues are integers, the assumptions of Lemma 1.2.1 are satisfied and we can prove the following theorem, which is also proved by De Winter in [19].

Theorem 3.1.1 *Let $\mathcal{S}$ be a partial geometry of order (s, t, α) and let θ be an automorphism of $\mathcal{S}$. If f_0 is the number of points fixed by θ and if f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then for some integer k there holds*

$$\mathrm{tr}(QM) = k(s+t+1-\alpha) + (1+s)(1+t) = (t+1)f_0 + f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (1+s)(1+t)J$, we have $(QM)J = (1+s)(1+t)J$, so $(1+s)(1+t)$ is an eigenvalue of QM. Because $m_1 = 1$, it follows that this eigenvalue of QM has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM with multiplicity m_0. For each divisor d of n, let ξ_d denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Then U_d is an integer by [31]. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = s+t+1-\alpha$, because of Lemma 1.2.1. Let a_d denote the multiplicity of $\xi_d(s+t+1-\alpha)$ as an eigenvalue of QM, with $d|n$, and ξ_d a primitive d^{th} root of unity. Then we have:

$$\mathrm{tr}(QM) = \sum_{d|n} a_d(s+t+1-\alpha)U_d + (1+s)(1+t),$$

or

$$\mathrm{tr}(QM) = k(s+t+1-\alpha) + (1+s)(1+t),$$

with k an integer.

Since the entry on the i^{th} row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\mathrm{tr}(QM) = (1+t)f_0 + f_1$. Hence

$$k(s+t+1-\alpha) + (1+s)(1+t) = (1+t)f_0 + f_1,$$

with k an integer. □

Corollary 3.1.2 *Let $\mathcal{S}$ be a partial geometry of order (s, t, α) and let θ be an automorphism of $\mathcal{S}$. If s, t and $\alpha - 1$ have a common divisor distinct from 1, then there exists at least one fixpoint or at least one point which is mapped to a point collinear to itself.*

Proof. Suppose that there are no fixpoints and no points which are mapped to a collinear point, hence $f_0 = f_1 = 0$. Because of the previous theorem, $k(s+t+1-\alpha)+(1+s)(1+t)$ has to be equal to 0. Hence $k(s+t+1-\alpha)+s+t+st = -1$. But because s, t and $\alpha-1$ have a common divisor distinct from 1, there exists an integer m which divides s, t and $\alpha - 1$. Hence m divides $k(s+t+1-\alpha)+s+t+st$, but m does not divide -1 and we have a contradiction. □

Note that if $\alpha = 1$ in the previous corollary, than we obtain Corollary 1.3.3.

Corollary 3.1.3 *Let $\mathcal{S}$ be a partial geometry of order (s, t, α) and let θ be an involution of $\mathcal{S}$. If s, t and $\alpha - 1$ have a common divisor distinct from 1, then there exists at least one fixpoint or at least one fixline.*

Proof. This follows immediately from the previous corollary because if there is a point x which is mapped to a point collinear to x by the involution θ, then the line xx^θ is a fixline. □

3.2 Dualities of symmetric partial geometries

We now have a look at the double of a symmetric partial geometry of order (t, t, α), which is a near octagon of order $(1, t; \alpha, 1)$.

If the matrix M of this near octagon is defined as before, then it has the following eigenvalues (cf. [16]):

eigenvalues of M	multiplicity
0	$m_0 = 1$
$2t + 2$	$m_1 = 1$
$1 + t$	$m_2 = \frac{2(2-\alpha)(t+\alpha)}{\alpha(t+2-\alpha)}$
$t + 1 + \sqrt{2t + 1 - \alpha}$	$m_3 = \frac{2(t+1)t}{\alpha(t+2-\alpha)}$
$t + 1 - \sqrt{2t + 1 - \alpha}$	$m_4 = \frac{2(t+1)t}{\alpha(t+2-\alpha)}$

Since these eigenvalues involve square roots, we will have to take Lemmas 1.2.2 and 1.2.3 into account.

Theorem 3.2.1 *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that $2t + 1 - \alpha$ is compatible with the order of θ. If f_0 is the number of points fixed by θ and f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then for some integers k_1, k_2 and k_3 there holds*

$$k_1(1+t) + k_2(1+t+\sqrt{2t+1-\alpha}) + k_3(1+t-\sqrt{2t+1-\alpha}) + 2(1+t) = (1+t)f_0 + f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = 2(1+t)J$, we have $(QM)J = 2(1+t)J$, so $2(1+t)$ is an eigenvalue of QM. Because $m_1 = 1$, it follows that this eigenvalue of QM has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM with multiplicity $m_0 = 1$. For each divisor d of n, let ξ_d denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Then U_d is an integer by [31]. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = 1+t+\sqrt{2t+1-\alpha}$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM with $|\varphi'| = 1+t-\sqrt{2t+1-\alpha}$, because of Lemmas 1.2.1 and 1.2.2. Let a_d denote the multiplicity of $\xi_d(1+t+\sqrt{2t+1-\alpha})$ and let b_d denote the multiplicity of $\xi_d(1+t-\sqrt{2t+1-\alpha})$ as eigenvalues of QM, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\operatorname{tr}(QM) = \sum_{d|n} a_d(1+t+\sqrt{2t+1-\alpha})U_d + \sum_{d|n} b_d(1+t-\sqrt{2t+1-\alpha})U_d + 2(1+t),$$

or

$$\operatorname{tr}(QM) = k_1(1+t+\sqrt{2t+1-\alpha}) + k_2(1+t-\sqrt{2t+1-\alpha}) + 2(1+t),$$

with k_1 and k_2 integers.
Since the entry on the i^{th} row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\operatorname{tr}(QM) = (1+t)f_0 + f_1$. Hence

$$k_1(1+t+\sqrt{2t+1-\alpha}) + k_2(1+t-\sqrt{2t+1-\alpha}) + 2(1+t) = (1+t)f_0 + f_1,$$

with k_1 and k_2 integers. □

Theorem 3.2.2 *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that $2t+1-\alpha$ is compatible with the order of θ. If f_0 is the number of points fixed by θ, f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$ and f_2 is the number of points for which $\mathrm{d}(x, x^\theta) = 4$, then for some integers k_1, k_2 and k_3 there holds*

$$k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2 = (2+t)(1+t)f_0 + (2+2t)f_1 + f_2.$$

Proof. Suppose that M, A and Q are defined as before. Suppose that θ has order n, so that $(QM^2)^n = Q^n M^{2n} = M^{2n}$. It follows that the eigenvalues of QM^2 are the eigenvalues of M^2 multiplied by the appropriate roots of unity. Since $M^2 J = (2(1+t))^2 J$, we have $(QM^2)J = (2(1+t))^2 J$, so $(2(1+t))^2$ is an eigenvalue of QM^2. By Lemma 1.2.5, as $m_1 = 1$ it follows that this eigenvalue of QM^2 has multiplicity 1. Further it is clear that 0 is an eigenvalue of QM^2 with multiplicity m_0.

Using Lemma 1.2.5 and arguing as in the proof of Theorem 1.4.4, we obtain

$$\mathrm{tr}(QM^2) = k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2,$$

with k_1, k_2 and k_3 the integers appearing in Theorem 3.2.1.

On the other hand we have

$$\begin{array}{rl} & M = A + (1+t)I \\ \Rightarrow & QM = QA + (1+t)Q \\ \Rightarrow & \mathrm{tr}(QM) = \mathrm{tr}(QA) + (1+t)\mathrm{tr}(Q) \\ \Rightarrow & (1+t)f_0 + f_1 = \mathrm{tr}(QA) + (1+t)f_0 \\ \Rightarrow & \mathrm{tr}(QA) = f_1. \end{array}$$

The matrix $A^2 = (a_{ij})$ is the matrix with $(1+t)$ along the main diagonal and on the other entries we have $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 4$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^2) = (1+t)f_0 + f_2$. It follows that

$$\begin{array}{rl} & \mathrm{tr}(QM^2) \\ = & \mathrm{tr}(Q(A+(1+t)I)^2) \\ = & \mathrm{tr}(QA^2) + 2(1+t)\mathrm{tr}(QA) + (1+t)^2\mathrm{tr}(Q) \\ = & (1+t)f_0 + f_2 + 2(1+t)f_1 + (1+t)^2 f_0 \\ = & (2+t)(1+t)f_0 + 2(1+t)f_1 + f_2. \end{array}$$

□

Theorem 3.2.3 *Let $\mathcal{S}$ be a near octagon of order $(1, t; \alpha, 1)$ and let θ be a nontrivial automorphism of $\mathcal{S}$. Assume that $2t+1-\alpha$ is compatible with the order of θ. If f_0 is the number of points fixed by θ, f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, f_2 is the number of points for which $\mathrm{d}(x, x^\theta) = 4$ and f_3 is the number of points for which $\mathrm{d}(x, x^\theta) = 6$, then for some integers k_1, k_2 and k_3 holds*

$$k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + ((1+s)(1+t))^3 = (3(1+t)^2+(1+t)^3)f_0 + (1+2t+3(1+t)^2)f_1 + 3(1+t)f_2 + \alpha f_3.$$

Proof. Suppose that M, A and Q are defined as before. In the same way as in the proofs of Theorems 3.2.1 and 3.2.2 we can prove that $\mathrm{tr}(QM^3) = k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + (2(1+t))^3$, with k_1, k_2 and k_3 the same integers as in these proofs. On the other hand, we can calculate that $A^3 = (a_{ij})$ is the matrix with 0 along the main diagonal while on the other entries we have $a_{ij} = 1+2t$ if $x_i \sim x_j$, $a_{ij} = \alpha$ if $\mathrm{d}(x_i, x_j) = 6$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^3) = (1+2t)f_1 + \alpha f_3$. Because of the proof of Theorem 3.2.2 we know that $\mathrm{tr}(QA^2) = (1+t)f_0 + f_2$, $\mathrm{tr}(QA) = f_1$ and $\mathrm{tr}(Q) = f_0$. Hence

$$\begin{aligned}
& \mathrm{tr}(QM^3) \\
= \; & \mathrm{tr}(Q(A+(1+t)I)^3) \\
= \; & \mathrm{tr}(QA^3) + 3(1+t)\mathrm{tr}(QA^2) + 3(1+t)^2\mathrm{tr}(QA) + (1+t)^3\mathrm{tr}(Q) \\
= \; & (1+2t)f_1 + \alpha f_3 + 3(1+t)((1+t)f_0 + f_2) + 3(1+t)^2 f_1 + (1+t)^3 f_0 \\
= \; & (3(1+t)^2 + (1+t)^3)f_0 + ((1+2t) + 3(1+t)^2)f_1 + 3(1+t)f_2 + \alpha f_3.
\end{aligned}$$

□

As already mentioned in the proofs, the integers k_1, k_2 and k_3 in Theorems 3.2.1, 3.2.2 and 3.2.3 are the same by Lemma 1.2.5.

Suppose now that, under the same assumptions of the previous theorems, θ is a duality in the underlying partial geometry, then we know that $f_0 = 0$ and $f_2 = 0$. Because of Theorems 3.2.1, 3.2.2 and 3.2.3 , we have the following equations:

$$\begin{cases}
k_1(1+t) + k_2(1+t+\sqrt{2t+1-\alpha}) + k_3(1+t-\sqrt{2t+1-\alpha}) + 2(1+t) = f_1, \\
k_1(1+t)^2 + k_2(1+t+\sqrt{2t+1-\alpha})^2 + k_3(1+t-\sqrt{2t+1-\alpha})^2 + (2(1+t))^2 = \\
\qquad (2+2t)f_1, \\
k_1(1+t)^3 + k_2(1+t+\sqrt{2t+1-\alpha})^3 + k_3(1+t-\sqrt{2t+1-\alpha})^3 + (2(1+t))^3 = \\
\qquad (1+2t+3(1+t)^2)f_1 + \alpha f_3.
\end{cases}$$

Because f_0 and f_2 are 0, we know that $f_1 + f_3 = \frac{2(t+1)(\alpha+t^2)}{\alpha}$. Hence:

$$\begin{cases}
k_1 = 0, \\
k_2 = \frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}}, \\
k_3 = -\frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}}, \\
f_3 = \frac{2(t+1)(\alpha+t^2)}{\alpha} - f_1.
\end{cases}$$

So $\frac{-2(t+1)+f_1}{2\sqrt{2t+1-\alpha}}$ has to be an integer. In the case that $2t+1-\alpha$ is not a square, this only holds if $f_1 = 2(t+1)$. Suppose that $2t+1-\alpha$ is a square, then $f_1 - 2(t+1)$ should be a multiple of $2\sqrt{2t+1-\alpha}$. If α is odd, then $f_1 \equiv 1+\alpha \mod 2\sqrt{2t+1-\alpha}$. If α is even, then $f_1 \equiv 1+\alpha+\sqrt{2t+1-\alpha} \mod 2\sqrt{2t+1-\alpha}$.

Corollary 3.2.4 *If θ is a duality of a partial geometry of order (t,t,α), with $2t+1-\alpha$ not a square and compatible with the order of θ, then θ has $1+t$ absolute points and $1+t$ absolute lines, and there are $(1+t)t^2/\alpha$ points which are mapped to a line at distance 3 and $(1+t)t^2/\alpha$ lines which are mapped to a point at distance 3.*

Corollary 3.2.5 *Suppose that θ is a duality of a partial geometry of order (t,t,α), with $2t+1-\alpha$ a square. If α is odd, then it has $(1+\alpha)/2 \mod \sqrt{2t+1-\alpha}$ absolute points and equally many absolute lines. If α is even, then it has $(1+\alpha+\sqrt{2t+1-\alpha})/2 \mod \sqrt{2t+1-\alpha}$ absolute points and equally many absolute lines.*

For example, every duality of the sporadic partial geometry of Van Lint & Schrijver $((t,t,\alpha) = (5,5,2))$ must always have $0 \mod 3$ absolute points (here, $\sqrt{2t+1-\alpha} = 3$). Note that the standard polarity (see Subsection 0.7.3) has exactly $6 = 1+t$ absolute points.

3.3 An alternative description of some partial geometries arising from Thas 1974 maximal arcs

Let C be a Thas 1974 maximal arc, and let $\mathsf{pg}(C)$ be the corresponding partial geometry; see Subsection 0.7.2. Recall that $\mathsf{pg}(C)$ has order $(2^{2m}-2^m, 2^{2m}-2^m, 2^{2m}-2^{m+1}+1)$. Recall also that C is constructed using an ovoid $\mathcal{O}$ and a 1-spread $\mathcal{R}$ of $\mathsf{PG}(3,2^m)$, $m>0$, such that each line of $\mathcal{R}$ has one and only one point in common with $\mathcal{O}$. An interesting example of this situation occurs when $\mathcal{R}$ is a regular spread (so there arises a Desarguesian projective plane of order 2^{2m}) and the ovoid is a Suzuki-Tits ovoid (hence the maximal arc is not a Denniston maximal arc; see [47]). In the following we determine the isomorphism classes of such maximal arcs and of the corresponding partial geometries. We also determine the full automorphism groups and correlation groups of these structures.

In order to do so, and in particular in order to prove that the partial geometries are self-dual, we first give an alternative description of the maximal arcs in a more homogeneous setting.

Consider the projective space $\mathsf{PG}(5,q)$ and suppose that we have a regular spread $\mathcal{S}$ of lines in this space. It is well known — and easy to see — that the lines of this spread can be considered as the points of a projective plane $\mathsf{PG}(2,q^2)$ while the 3-spaces of $\mathsf{PG}(5,q)$ containing q^2+1 spread lines are the lines of this projective plane. Fix such a 3-space $\mathsf{PG}(3,q)$ and denote by L_∞ the corresponding line of $\mathsf{PG}(2,q^2)$. Let $\mathcal{O}$ be an ovoid in $\mathsf{PG}(3,q)$ such that every point of $\mathcal{O}$ is incident with a unique line of $\mathcal{S}$. Take a line L of $\mathcal{S}$ outside $\mathsf{PG}(3,q)$ and a point x incident with L. Let $\mathsf{PG}(4,q)$ be the hyperplane generated by $\mathsf{PG}(3,q)$ and x. Then there is a bijective correspondence β between the points of $\mathsf{PG}(4,q)\setminus\mathsf{PG}(3,q)$ and the lines of $\mathcal{S}$ not in $\mathsf{PG}(3,q)$ given by containment. It is also obvious that a 3-space distinct from $\mathsf{PG}(3,q)$ containing q^2+1 spread lines intersects $\mathsf{PG}(4,q)$ in a plane π which on its turn intersects $\mathsf{PG}(3,q)$ in a member of $\mathcal{S}$. Hence the bijection β described above defines an isomorphism between the two models of $\mathsf{PG}(2,q^2)$.

Using β, we now see that in $\mathsf{PG}(5,q)$, the spread lines corresponding to points of the Thas 1974 maximal arc C defined by $\mathcal{O}$ and x are the elements of $\mathcal{S}$ not in $\mathsf{PG}(3,q)$ that meet a line xp in a point, where $p\in\mathcal{O}$.

3.4 Collineations and dualities of the partial geometry $\mathsf{pg}(C)$

3.4.1 Duality problem

In this section we show that the partial geometry $\mathsf{pg}(C)$, with C a Thas 1974 maximal arc in the Desarguesian projective plane $\mathsf{PG}(2,q^2)$, is self-dual.

Note that, for a given maximal arc C in any projective plane, the set of external lines of C is a maximal arc C^* in the dual projective plane, and it has the complementary parameters, i.e., if C is a maximal $\{qn-q+n,n\}$-arc, then C^* is a (dual) $\{qh-q+h,h\}$-arc, with $nh=q$. In the case of a Thas 1974 maximal arc considered above, we see that $n=h=2^m$.

So, in order to prove that the partial geometry related to a Thas 1974 maximal arc is self-dual, it suffices to show that the corresponding Thas 1974 maximal arc is *"self-dual"*, i.e., a Thas 1974 maximal arc C is projectively equivalent with the set C^* of external lines in the dual projective plane.

So let C be a Thas 1974 maximal arc in $\mathsf{PG}(2,q^2)$, constructed as above using the ovoid $\mathcal{O}$. First of all, we remark that the set of tangent planes of $\mathcal{O}$ is an ovoid $\mathcal{O}^*$ in the dual of $\mathsf{PG}(3,q)$. Indeed, the set of tangent lines of $\mathcal{O}$ is the line set of a symplectic generalized quadrangle $\mathsf{W}(q)$, which arises from a (symplectic) polarity ρ of $\mathsf{PG}(3,q)$. This symplectic polarity maps each point of $\mathsf{PG}(3,q)$ onto the plane spanned by the lines of $\mathsf{W}(q)$ through x. Hence it maps each point of $\mathcal{O}$ onto its tangent plane. Now it is also clear that $\mathcal{O}$ and $\mathcal{O}^*$ are isomorphic.

Next we consider the following construction of C. We dualize in $\mathsf{PG}(5,q)$ the construction of $\mathsf{PG}(2,q^2)$ outlined above. The line L not in $\mathsf{PG}(3,q)$ of the spread plays the role of the space $\mathsf{PG}(3,q)$; the ovoid $\mathcal{O}$, as a set of points in $\mathsf{PG}(3,q)$ is replaced by the set of hyperplanes (which we will call the *dual ovoid* in the sequel) spanned by L and the tangent planes to $\mathcal{O}$ in $\mathsf{PG}(3,q)$. The space $\mathsf{PG}(3,q)$ plays the role of L. The point x plays the role of the hyperplane X generated by x and $\mathsf{PG}(3,q)$. The spread lines in $\mathsf{PG}(3,q)$ and the 3-spaces containing L and q^2+1 spread lines are also interchanged. Let H be an element of the dual ovoid. We claim that H contains a unique 3-space K containing L and q^2+1 spread lines. Indeed, K is the 3-space generated by L and the spread line incident with the point of $\mathcal{O}$ obtained by intersecting the tangent plane of $\mathcal{O}$ corresponding to H with $\mathcal{O}$. Now, interpreting the Thas 1974 maximal arc in this dual setting in the $\mathsf{PG}(5,q)$-model of $\mathsf{PG}(2,q^2)$, this maximal arc consists of those 3-spaces S containing q^2+1 spread lines and contained in a hyperplane which contains $\langle x,\pi\rangle$ but not L, where π is a tangent plane of $\mathcal{O}$. Then S contains the spread line T in π. It is clear that S has no point in common with the cone $x\mathcal{O}$, and hence defines a line of C^*.

Hence we have shown the following result.

Theorem 3.4.1 *Let C be a Thas 1974 maximal arc in $\mathsf{PG}(2,q^2)$, arising from an ovoid $\mathcal{O}$ in $\mathsf{PG}(3,q)$ by considering the points of the cone $x\mathcal{O}$ not in $\mathsf{PG}(3,q)$. Then C is isomorphic to its dual C^*, and there is a duality of $\mathsf{PG}(2,q^2)$ that interchanges the point x with the line $L_\infty = \mathsf{PG}(3,q)$. In particular, the partial geometry which arises from this maximal arc is self-dual.*

We will now apply the Benson-type formulas to this example. We have a partial geometry of order (s,t,α), with (cf. [47]):

$$s = t = 2^{2m} - 2^m \text{ and } \alpha = 2^{2m} - 2^{m+1} + 1.$$

And the maximal arc is a $\{2^{3m} - 2^{2m} + 2^m, 2^m\}$-arc. Hence $2t + 1 - \alpha = 2^{2m}$, which is a square. In this case α is odd, hence

$$\begin{aligned} f_1 &\equiv 1+\alpha \mod 2^{m+1} \\ &\equiv 1+2^{2m}-2^{m+1}+1 \mod 2^{m+1} \\ &\equiv 2 \mod 2^{m+1}. \end{aligned}$$

We can conclude that if we have a duality in this partial geometry, then there will be at least one absolute point and one absolute line.

3.4.2 Automorphism problem

Consider the construction which we described in Section 0.7.2. So we have a projective plane $\mathsf{PG}(2,q^2)$ and a maximal arc C. Consider the partial geometry which arises from this arc and a collineation of this partial geometry. Now we will have a look at this collineation in the projective plane. The points outside the maximal arc are permuted and also the lines which intersect the maximal arc are permuted. Consider a line outside the maximal arc. This is a set of q^2+1 points, with the condition that any two of them are non-collinear in the partial geometry. Hence this line is mapped to a set B of q^2+1 mutual not collinear points. Consider a point z of B. From the foregoing, we deduce that every line containing z which intersects C non-trivially is a tangent line to B. Hence every point of the maximal arc is a nucleus of B and because of Theorem 13.43 in [28] and the fact that $|C| > q-1$, B should be a line of the projective plane. Hence also the lines outside C and, by considering the dual maximal arc C^*, the points inside C are permuted and incidence is preserved (because we look at external lines as sets of points and at maximal arc points as sets of lines). We conclude that a collineation of the partial geometry, which arises from C, induces a collineation of the projective plane $\mathsf{PG}(2,q^2)$.

So we have the following result.

Theorem 3.4.2 *The collineation group of* $\mathsf{pg}(C)$ *is induced by the collineation group of* $\mathsf{PG}(2,q^2)$.

Remark 3.4.3 The previous theorem holds for all maximal arcs K in finite Desarguesian projective planes and their corresponding partial geometries $\mathsf{pg}(K)$, as defined in Subsection 0.7.2.

The previous theorem will be used in the next section to give a description of the complete correlation groups of the partial geometries arising from the Thas 1974 maximal arcs in $\mathsf{PG}(2,q^2)$ related to the Suzuki-Tits ovoids. But first we determine the isomorphism classes of such partial geometries.

At this point, we could refer to [27] to deduce that there are exactly two isomorphism classes of such geometries. However, it seems to us that the arguments in [27] tacitly assume that every collineation of $\mathsf{PG}(2, q^2)$ stabilizing C fixes x, without actually proving it or providing a reference. Hence we continue our proof.

3.4.3 Isomorphism problem

The arguments below will require that $m > 1$ (equivalently, $q > 2$). Henceforth, we assume $m > 1$. At the end we make a remark about the case $m = 1$.

Consider again the projective space $\mathsf{PG}(5, q)$ and a regular spread of lines in this space. Take a 3-space $\mathsf{PG}(3, q)$ containing $q^2 + 1$ spread lines in this 5-space and take a Suzuki-Tits ovoid $\mathcal{O}$ in this 3-space with the property that each point of $\mathcal{O}$ is on a unique spread line. The tangent lines to $\mathcal{O}$ form the lines of a symplectic quadrangle $\mathsf{W}(q)$ (cf.[28]). The lines of the spread which lie in this $\mathsf{PG}(3, q)$ are lines of $\mathsf{W}(q)$. Hence these lines form a spread S of $\mathsf{W}(q)$.

The Suzuki-Tits ovoid determines a unique polarity ρ of $\mathsf{W}(q)$ (see [57]; here we require $q > 2$). Hence we obtain a set of absolute lines which corresponds with ρ. This set of lines forms a (Lüneburg-Suzuki-Tits) spread T.

When we take the intersection of the spread S and the spread T, then, by [3], see also [18] we obtain two possibilities for the intersection number, namely

$$q + \sqrt{2q} + 1 \text{ and } q - \sqrt{2q} + 1.$$

It will turn out that the maximal arcs, which we obtain by taking a Suzuki-Tits ovoid, and for which we obtain $q + \sqrt{2q} + 1$ as intersection number are not isomorphic to those for which we obtain $q - \sqrt{2q} + 1$ as intersection number. To prove this, we determine the collineation groups of each maximal arc. Now, by [3], the subgroup of $\mathsf{PGL}_4(q)$ stabilizing S and T is dihedral of order $4|S \cap T|$; this has been reproved explicitly in [27] in a more "transparent" way. Taking into account all generalized homologies with center x and axis $\mathsf{PG}(3, q)$ in $\mathsf{PG}(4, q)$, one easily sees that the stabilizer of x and L_∞ inside the stabilizer of the maximal arc C in the group $\mathsf{PGL}_3(q^2) : 2$ (the extension of order 2 is due to the unique involution of $\mathsf{GF}(q^2)$, which is linear over $\mathsf{GF}(q)$ in $\mathsf{PG}(4, q)$), acting on $\mathsf{PG}(2, q^2)$ is a group of order $4|S \cap T|(q - 1)$ isomorphic to the direct product of the dihedral group of order $4|S \cap T|$ and a cyclic group of order $q - 1$. We now claim that every collineation stabilizing C must fix x.

We will first prove the following lemma.

Lemma 3.4.4 *Let $\mathcal{O}$ be a Suzuki-Tits ovoid in $\mathsf{PG}(3,q)$, $q>2$, and let π be a plane that intersects $\mathcal{O}$ in an oval O. Let T be the corresponding Lüneburg-Suzuki-Tits spread. If $q>8$ and $p\in O$, then $O\backslash\{p\}$ is no non degenerate conic minus a point. If $q=8$ and $p\in O\backslash\{p'\}$, with p' the point of O incident with the line of T in π, then $O\backslash\{p\}$ is no non degenerate conic minus a point.*

Proof. By [55] 7.6.13 we can choose the coordinates such that $\mathcal{O}=\{(1,0,0,0)\}\cup\{(a^{\theta+2}+aa'+a'^{\theta},1,a',a) : a,a'\in\mathbb{K}\}$, with θ a Tits automorphism, i.e. $(x^{\theta})^{\theta}=x^2$, $\forall x\in\mathsf{GF}(q)$. Since all plane intersections play the same role, we can choose the plane $X_3=0$. The oval O is the point set of the algebraic curve $\mathcal{C}' : X_0X_1^{\theta-1}+X_2^{\theta}=X_3=0$. Let $p\in O$, $q>8$ and assume, by way of contradiction that $O\backslash\{p\}$ is a non degenerate conic $\mathcal{C}$ minus a point. Then $\mathcal{C}$ and $\mathcal{C}'$ have at least q common points. As $q>2\theta$, by the Theorem of Bézout, $\mathcal{C}$ is a component of $\mathcal{C}'$. Hence O is a conic, contradiction. Next, let $q=8$, $p\in O$, $p\neq p'$, and assume, by way of contradiction, that $O\backslash\{p\}$ is a non degenerate conic $\mathcal{C}$ minus a point. Here $p'=(1,0,0,0)$ and $O\backslash\{p'\}$ is a conic $\mathcal{C}''$ minus a point. The conics $\mathcal{C}$ and $\mathcal{C}''$ have at least 7 points in common, so coincide. Hence O is a conic, a contradiction. $\square$

Note that the previous lemma is also true for the infinite case.

Now, all lines of $\mathsf{PG}(2,q^2)$ through x meet C in a Baer subline minus one point. Consider a point $z\in C$, $z\neq x$, and let π be a plane through z and through a line of $S\backslash T$. Put $C'=\pi\cap C$. Then the projection from x of C' onto $\mathsf{PG}(3,q)$ is a plane intersection of $\mathcal{O}$ minus a point of the Suzuki-Tits ovoid $\mathcal{O}$ satisfying the assumption of Lemma 3.4.4. Hence C' is not a Baer subline minus a point in $\mathsf{PG}(2,q^2)$. So, there are lines through every other point of C meeting C in a set different from a Baer subline minus one point. The claim that every collineation of $\mathsf{PG}(2,q^2)$ stabilizing C must fix x is proved. By Theorem 3.4.1 also L_∞ must be fixed by such a collineation. Now one sees that the full stabilizer of C is a group with a normal subgroup as described above, and corresponding factor group a group of order m (corresponding to the field automorphisms of $\mathsf{GF}(q)$).

Now one can quote [27] to conclude, but since we've come that far, we finish for completeness' sake.

The previous not only shows that the order of the full collineation group of C, and hence also of $\mathsf{pg}(C)$, is equal to $4m|S\cap T|(q-1)$, $q=2^m$, but it also shows that the two partial geometries related to the two different intersections are not isomorphic.

At last we show that two partial geometries $\mathsf{pg}(C)$ and $\mathsf{pg}(C')$ related to two maximal arcs C and C' corresponding to respective Suzuki-Tits ovoids $\mathcal{O}$ and $\mathcal{O}'$, for which the corresponding respective spreads T and T' satisfy $|S\cap T|=|S\cap T'|$, are isomorphic.

First we claim that for a given intersection $S \cap T$ (with T a Lüneburg-Suzuki-Tits spread), T is the only Lüneburg-Suzuki-Tits spread intersecting S in $S \cap T$. Indeed, we count the number of all possible intersections of S with some Lüneburg-Suzuki-Tits spread that occur. As above, it follows from [3] (see also [18]) that, for $\epsilon \in \{+1, -1\}$, the intersection of size $q + \epsilon\sqrt{2q} + 1$ occurs at least

$$\frac{|\mathsf{PGL}_2(q^2)|}{2(q + \epsilon\sqrt{2q} + 1)} = \frac{(q^2+1)q^2(q^2-1)}{2(q + \epsilon\sqrt{2q} + 1)} = \frac{1}{2}(q - \epsilon\sqrt{2q} + 1)(q^2(q^2-1)$$

times. Hence, in total, we have at least $(q+1)q^2(q^2-1)$ possible intersections that occur. But this is equal to the index of the Suzuki group in the symplectic group, namely

$$\frac{q^4(q^4-1)(q^2-1)}{(q^2+1)q^2(q-1)},$$

which is precisely the number of Lüneburg-Suzuki-Tits spreads. Our claim follows.

Now since every two intersections of the same size can be mapped onto each other, while preserving S, and there are unique Suzuki-Tits ovoids corresponding with them, we conclude that the corresponding maximal arcs are isomorphic.

Hence we have shown:

Theorem 3.4.5 *There are exactly two isomorphism classes of partial geometries* $\mathsf{pg}(C)$ *in* $\mathsf{PG}(2, q^2)$*, with* $q = 2^m$*, where* C *is a Thas maximal arc in* $\mathsf{PG}(2, q^2)$ *corresponding to a Suzuki-Tits ovoid (with* $m > 1$ *odd). Each such partial geometry is self-dual and each collineation and duality of* $\mathsf{pg}(C)$ *is induced by a collineation or duality of the projective plane* $\mathsf{PG}(2, q^2)$*. The size of the full automorphism group is* $8m(2^m + \epsilon 2^{\frac{m+1}{2}} + 1)(2^m - 1)$*, with* $\epsilon \in \{+1, -1\}$*.*

Remark 3.4.6 If $q = 2$, then any maximal arc in $\mathsf{PG}(2, 4)$ is a hyperoval obtained by adding the nucleus to a conic. The corresponding partial geometry is the unique generalized quadrangle of order $(2, 2)$, which is isomorphic to $\mathsf{W}(2)$. Also in this case, the full collineation group and correlation group are induced by the collineation and correlation group of $\mathsf{PG}(2, 4)$, see for instance [43].

4

Partial quadrangles and near decagons

In this chapter we apply the techniques developed in Chapter 1 to a last class of geometries, namely, the partial quadrangles. This class fits very well into the series of geometries we consider since in case that there are as many points on a line as lines through a point (the symmetric case), the double is a near 10-gon (and in the previous chapters we encountered near 6-gons and near 8-gons). However, whereas there were many examples of designs (and still a lot of symmetric designs) and fewer but still a reasonable number of partial geometries (few symmetric ones, though), there are very few examples of partial quadrangles, and one does not even know of any thick symmetric partial quadrangle which is not a generalized quadrangle. So this last chapter of Part I is merely here for completeness, and we have no applications. Perhaps our results can be used to show nonexistence of certain collineations or dualities in hypothetical partial quadrangles appearing in arguments elsewehere. Let us also remark that Van Maldeghem [56] deduced a restriction on the parameters of a self-polar partial quadrangle which is not a generalized quadrangle using the incidence matrix (and this yields a restriction on the existence of polarities in partial quadrangles that are not generalized quadrangles).

As usual we repeat some proofs in the present setting just to allow the readers to skip the previous chapters and focus on this one.

4.1 A Benson-type theorem for partial quadrangles

We recall Lemmas 1.2.1 and 1.2.5 from Chapter 1 and we use the following notation analogous to the notation of the previous chapters.

Let A be an adjacency matrix of the point graph of a partial quadrangle Γ of order (s,t,μ) with v points, let $M = A + (t+1)I$, let θ be an automorphism of Γ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise. Recall that $QM = MQ$; see Section 1.2.

We now introduce some further notation. Suppose that $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ is a partial quadrangle of order (s,t,μ). Recall that we use the notion of collinearity only for distinct points. Let D be an incidence matrix of Γ. Then $M := DD^T = A + (t+1)I$, where A is an adjacency matrix of the point graph of Γ. Let θ be an automorphism of Γ of order n and let $Q = (q_{ij})$ be the $v \times v$ matrix with $q_{ij} = 1$ if $x_i^\theta = x_j$ and $q_{ij} = 0$ otherwise; so Q is a permutation matrix. Because $M = A + (t+1)I$, the eigenvalues of M are as follows:

eigenvalues of M	multiplicity
$(s+1)(t+1)$	$m_0 = 1$
$\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}$	m_1
$\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}$	m_2

Theorem 4.1.1 *Let Γ be a partial quadrangle of order (s,t,μ), with $(s,t) \neq (1,1)$, and let θ be an automorphism of Γ. If f_0 is the number of points fixed by θ and if f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$, then for some integers k_1 and k_2 there holds*

$$k_1\left(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}\right) + k_2\left(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}\right) + (s+1)(t+1) = (t+1)f_0 + f_1.$$

Proof. Suppose that θ has order n, so that $(QM)^n = Q^n M^n = M^n$. It follows that the eigenvalues of QM are the eigenvalues of M multiplied by the appropriate roots of unity. Let J be the $v \times v$ matrix with all entries equal to 1. Since $MJ = (s+1)(t+1)J$, we have $(QM)J = (s+1)(t+1)J$, so $(s+1)(t+1)$ is an eigenvalue of QM. Because $m_0 = 1$, it follows that this eigenvalue of QM has multiplicity 1. For each divisor d of n, let ξ_d

denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Now U_d is the coefficient of the term of the second largest degree of the corresponding cyclotomic polynomial $\Phi_n(x)$. And since $\Phi_n(x) \in \mathbb{Z}[x]$, by [22], U_d is an integer. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM with $|\varphi| = \frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM with $|\varphi'| = \frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}$, because of Lemma 1.2.1, and the fact that for $(s,t) \neq (1,1)$ the number $(s-\mu+1)^2+4st$ is a perfect square (see Section 0.8). Let a_d denote the multiplicity of $\xi_d(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})$ and let b_d denote the multiplicity of $\xi_d(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})$ as eigenvalues of QM, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\operatorname{tr}(QM) = \sum_{d|n} a_d(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})U_d +$$
$$\sum_{d|n} b_d(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})U_d + (s+1)(t+1),$$

or

$$\operatorname{tr}(QM) = k_1(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}) + k_2(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}) + (s+1)(t+1),$$

with k_1 and k_2 integers.

Since the entry on the i^{th} row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\operatorname{tr}(QM) = (t+1)f_0 + f_1$. Hence

$$k_1(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}) + k_2(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}) + (s+1)(t+1) = (t+1)f_0 + f_1,$$

with k_1 and k_2 integers. $\square$

Theorem 4.1.2 *Let Γ be a partial quadrangle of order (s,t,μ), with $(s,t) \neq (1,1)$, and let θ be an automorphism of Γ. If f_0 is the number of points fixed by θ, f_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$ and f_2 is the number of points for which $\mathrm{d}(x, x^\theta) = 4$, then for some integers k_1 and k_2 there holds*

$$k_1(\tfrac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})^2 + k_2(\tfrac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})^2 + ((s+1)(t+1))^2 = (s+t+1)(t+1)f_0 + (2(1+t)+(s-1))f_1 + \mu f_2.$$

Proof. Suppose that M, A and Q are defined as before. Suppose that θ has order n, so that $(QM^2)^n = Q^n M^{2n} = M^{2n}$. It follows that the eigenvalues of QM^2 are the eigenvalues of M^2 multiplied by the appropriate roots of unity. Since $M^2J = ((s+1)(1+t))^2J$, we have $(QM^2)J = ((s+1)(t+1))^2J$, so $((s+1)(t+1))^2$ is an eigenvalue of QM^2. Because $m_0 = 1$ and it follows that this eigenvalue of QM^2 has multiplicity 1. For each divisor d of n, let ξ_d again denote a primitive d^{th} root of unity, and put $U_d = \sum \xi_d^i$, where the summation is over those integers $i \in \{1, 2, \ldots, d-1\}$ that are relatively prime to d. Now U_d is the coefficient of the term of the second largest degree of the corresponding cyclotomic polynomial $\Phi_n(x)$. And since $\Phi_n(x) \in \mathbb{Z}[x]$, by [22], U_d is an integer. For each divisor d of n, the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ of QM^2 with $|\varphi| = (\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})^2$ and also the primitive d^{th} roots of unity all contribute the same number of times to the eigenvalues φ' of QM^2 with $|\varphi'| = (\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})^2$, because of Lemma 1.2.1 and the fact that for $(s,t) \neq (1,1)$ the eigenvalues are integers. Let a_d denote the multiplicity of $\xi_d(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})^2$ and let b_d denote the multiplicity of $\xi_d(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})^2$ as eigenvalues of QM^2, with $d|n$ and ξ_d a primitive d^{th} root of unity. Then we have:

$$\operatorname{tr}(QM^2) = \sum_{d|n} a_d(\frac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})^2 U_d + \sum_{d|n} b_d(\frac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})^2 U_d + ((s+1)(t+1))^2,$$

or

$$\operatorname{tr}(QM^2) = k_1(\tfrac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2})^2 + k_2(\tfrac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2})^2 + ((s+1)(t+1))^2,$$

with k_1 and k_2 integers. On the other hand we have

$$\begin{aligned}
& M = A + (1+t)I \\
\Rightarrow\quad & QM = QA + (1+t)Q \\
\Rightarrow\quad & \operatorname{tr}(QM) = \operatorname{tr}(QA) + (1+t)\operatorname{tr}(Q) \\
\Rightarrow\quad & (1+t)f_0 + f_1 = \operatorname{tr}(QA) + (1+t)f_0 \\
\Rightarrow\quad & \operatorname{tr}(QA) = f_1.
\end{aligned}$$

The matrix $A^2 = (a_{ij})$ is the matrix with $s(t+1)$ along the main diagonal and on the other entries we have $a_{ij} = s-1$ if $\mathrm{d}(x_i, x_j) = 2$, $a_{ij} = \mu$ if $\mathrm{d}(x_i, x_j) = 4$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^2) = s(t+1)f_0 + (s-1)f_1 + \mu f_2$. It follows that

$$\begin{aligned}
& \mathrm{tr}(QM^2) \\
= \; & \mathrm{tr}(Q(A + (1+t)I)^2) \\
= \; & \mathrm{tr}(QA^2) + 2(1+t)\mathrm{tr}(QA) + (1+t)^2\mathrm{tr}(Q) \\
= \; & s(t+1)f_0 + (s-1)f_1 + \mu f_2 + 2(1+t)f_1 + (1+t)^2 f_0 \\
= \; & (s+t+1)(1+t)f_0 + (2(1+t) + (s-1))f_1 + \mu f_2.
\end{aligned}$$

□

Remark 4.1.3 Note that in the previous theorem, f_2 can be written as $v - f_0 - f_1$, with v the total number of points. This gives an additional relation, which we shall use below in the examples.

We can apply the results of this section to some specific examples.

4.2 Some examples

In the following, we give some examples of how one can use the foregoing formulae in some specific situations. Of course, in many cases one knows explicitly the collineation group and so one could also prove the claims below using the explicit form of the collineation group. However, our examples just intend to show that some observations follow rather easily from the above. We certainly do not pretend that all observations are new, but an explicit check without the above theory varies from easy to tedious.

In each example, we eliminate f_0, f_1, f_2 from the formulae of Theorems 4.1.1 and 4.1.2, and of Remark 4.1.3.

The ordinary pentagon

The ordinary pentagon is the only geometry considered in this thesis that is not isomorphic or dual to the double of another geometry and which has non-integer eigenvalues. Hence, in principle, its collineations should not satisfy the conclusion of Theorem 4.1.1. Of course, if θ is an involution, then it does (with $k_1 = k_2 = 0$, $f_0 = 1$ and $f_1 = 2$) because every prime is compatible with 2. But let us consider a collineation of order 5, and note that 5 is not compatible with itself. If the conclusion of Theorem 4.1.2 remained true, then we would have $k_1 = k_2$ and $3k_1 = 2f_0 + f_1 - 4 = f_1 - 4$, with $f_1 \in \{0, 5\}$. This is clearly impossible for integer k_1.

The Petersen graph

In this case $(v, s, t, \mu) = (10, 1, 2, 1)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = k_1 - 2k_2 + 3, \\ f_2 = -2k_1 + k_2 + 6, \end{cases}$$

with k_1 and k_2 integers. Since in every of the above equalities, the coefficient of either k_1 or k_2 equals 1, we cannot obtain a restriction by considering some equality modulo some natural number. However, we can picture the above equations with respect to an orthonormal (k_1, k_2)-coordinate system, see the picture below. Here, we can for instance see that, since the lines with equations $f_0 = 0$ and $f_2 = 0$ do not meet in a point with integer coordinates, there are no collineations mapping every point to a collinear point. Similarly, there is no collineation mapping every point to an "opposite" point.

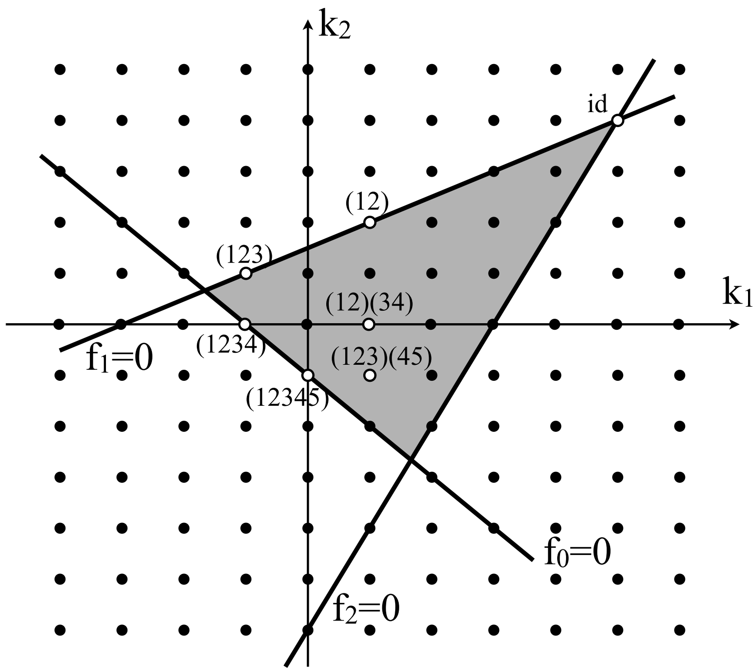
In the picture below we have marked the lattice points that really occur by writing a representative of the corresponding conjugacy class of collineations in $\mathsf{Sym}(5)$ using the representation of the Petersen graph as the set of unordered pairs of the set $\{1, 2, 3, 4, 5\}$ with adjacency given by being disjoint.

The Clebsch graph

In this case $(v, s, t, \mu) = (16, 1, 4, 2)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = k_1 - 3k_2 + 5, \\ f_2 = -2k_1 + 2k_2 + 10, \end{cases}$$

with k_1 and k_2 integers. Here we can only deduce that f_2 is always even, so that every collineation must map an even number of vertices to distinct non-adjacent vertices.



Hoffman-Singleton graph

In this case $(v, s, t, \mu) = (50, 1, 6, 1)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 2k_1 - 3k_2 + 7, \\ f_2 = -3k_1 + 2k_2 + 42, \end{cases}$$

with k_1 and k_2 integers. Here we can draw some conclusions modulo 5. Indeed, $f_1 \equiv f_2 \equiv 2f_0 \mod 5$.

The Gewirtz graph

In this case $(v, s, t, \mu) = (56, 1, 9, 2)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 2k_1 - 4k_2 + 10, \\ f_2 = -3k_1 + 3k_2 + 45, \end{cases}$$

with k_1 and k_2 integers. Here we easily see that f_1 always must be even and that f_2 must always be a multiple of 3. In particular, no collineation maps every vertex to a distinct non-adjacent vertex.

The Higman-Sims graphs

In these cases $(v, s, t, \mu) = (77, 1, 15, 4)$ or $(100, 1, 21, 6)$ and we obtain respectively

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 2k_1 - 6k_2 + 16, \\ f_2 = -3k_1 + 5k_2 + 60, \end{cases}$$

or

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 2k_1 - 8k_2 + 22, \\ f_2 = -3k_1 + 7k_2 + 77, \end{cases}$$

with k_1 and k_2 integers.

In the first case we deduce that f_1 must be even. In particular no collineation maps every vertex to an adjacent vertex. Also no collineation maps every vertex to a distinct non-adjacent vertex since this would require $k_1 = -\frac{11}{4}$ and $k_2 = \frac{7}{4}$, which also follows from the observation that $f_0 \equiv f_2 + 1 \mod 4$.

In the second case we again deduce that f_1 must be even and $f_0 \equiv f_2 \mod 2$.

Remark 4.2.1 We observe that in the previous expressions for f_0, f_1 and f_2 in terms of k_1 and k_2 the independent terms are equal to 1, $t + 1$ and $v - t - 2$. This is not a coincidence, since it follows from the general form of these equations, which one can easily compute.

Remark 4.2.2 The examples above are all strongly regular graphs. The same technique can be used to derive similar results for any other strongly regular graph with integer eigenvalues. We have not done so here since this lies slightly outside the scope of this thesis and would take us too far away from the geometries and the buildings of classical type.

Partial quadrangles arising from generalized quadrangles of order (q, q^2)

Here, the partial quadrangle is derived from a generalized quadrangle Γ of order (q, q^2) as explained in Section 0.8. In this case $(v, s, t, \mu) = (q^4, q-1, q^2, q(q-1))$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = (q-1)k_1 - (q^2 - q + 1)k_2 + q^3 - q^2 + q - 1, \\ f_2 = -qk_1 + (q^2 - q)k_2 + q^4 - q^3 + q^2 - q, \end{cases}$$

for certain integers k_1 and k_2. Combining the second and the first equation we obtain $f_1 = (q-1)f_0 - q^2k_2 + q^3 - q^2$, which implies $f_1 \equiv (q-1)f_0 \mod q^2$. This gives additional information independent from the original Benson formula for generalized quadrangles (see Theorem 1.3.1) applied to Γ. Indeed, the latter does not give information about only the points opposite the point of the quadrangle used to define the partial quadrangle.

Partial quadrangles arising from caps

For the partial quadrangle arising from a suitable 11-cap in $\mathsf{PG}(4,3)$ we have $(v, s, t, \mu) = (243, 2, 10, 2)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 4k_1 - 5k_2 + 22, \\ f_2 = -5k_1 + 4k_2 + 220, \end{cases}$$

with k_1 and k_2 integers. Here we see that $f_1 \equiv f_2 \mod 9$. An other observation is $f_1 \equiv 4f_0 \mod 9$.

For the partial quadrangle arising from the unique 56-cap in $\mathsf{PG}(5,3)$ we have $(v, s, t, \mu) = (729, 2, 55, 20)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 4k_1 - 23k_2 + 112, \\ f_2 = -5k_1 + 22k_2 + 616, \end{cases}$$

with k_1 and k_2 integers. We again see that $f_1 \equiv f_2 \mod 9$. Also, we observe $f_1 \equiv 4f_0 \mod 27$.

For the partial quadrangle arising from a 78-cap in $\mathsf{PG}(5,4)$ we have $(v, s, t, \mu) = (4096, 3, 77, 14)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 10k_1 - 22k_2 + 234, \\ f_2 = -11k_1 + 21k_2 + 3861, \end{cases}$$

with k_1 and k_2 integers. We see that f_1 is always even and that $f_1 \equiv 2f_2 \mod 32$.

For the partial quadrangle arising from some hypothetical 430-cap in $\mathsf{PG}(6,4)$ we have $(v,s,t,\mu) = (16384, 3, 429, 110)$ and we obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = 10k_1 - 118k_2 + 1290, \\ f_2 = -11k_1 + 117k_2 + 15093, \end{cases}$$

with k_1 and k_2 integers. We again see that f_1 is always even and that $f_1 \equiv 2f_2 \mod 32$.

Partial quadrangles arising from hemisystems

Here we have the partial quadrangle arising from a hemisystem in a generalized quadrangle of order (q,q^2), and so $(v,s,t,\mu) = (\frac{(1+q)(1+q^3)}{2}, \frac{q-1}{2}, q^2, \frac{(q-1)^2}{2})$. We obtain

$$\begin{cases} f_0 = k_1 + k_2 + 1, \\ f_1 = (q-1)k_1 - \frac{1}{2}(q^2 - q + 2)k_2 + \frac{1}{2}(q^3 - q^2 + q - 1), \\ f_2 = -qk_1 + \frac{1}{2}q(q-1)k_2 + \frac{1}{2}q^2(q^2+1), \end{cases}$$

with k_1 and k_2 integers. We see that if q is odd then $f_2 \equiv 0 \mod q$ and $f_0 + f_1 \equiv \frac{q+1}{2} \mod q$, which implies in particular that θ must map some point to itself or to a collinear point.

4.3 Dualities of partial quadrangles

We recall that the double of a partial quadrangle of order (t,t,μ) is a near decagon of order $(1,t;\mu,1,1)$ see Section 0.8.2.

If the matrix M of this near decagon is defined as before, then it has the following eigenvalues:

eigenvalues of M	multiplicity
$2(t+1)$	$m_0 = 1$
0	m_1
$(t+1)+\frac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu}$	m_2
$(t+1)-\frac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu}$	m_3
$(t+1)+\frac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu}$	m_4
$(t+1)-\frac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu}$	m_5

Similarly as in the previous chapters, we can now prove some formulae for the number of points of a near decagon mapped to a point at certain distance by a fixed collineation.

Theorem 4.3.1 *Let $\mathcal{S}$ be a near decagon of order $(1,t;\mu,1,1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that, for $\epsilon=-1,1$, the number $2+2\epsilon\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ is compatible with the order of θ. If g_0 is the number of points fixed by θ and g_1 is the number of points x for which $x^\theta \neq x \sim x^\theta$ then for some integers k_1, k_2, k_3 and k_4 there holds*

$$\begin{gathered}
k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+\\
k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+\\
k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+\\
k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+2(1+t)=\\
(t+1)g_0+g_1.
\end{gathered}$$

Proof. This proof is completely similar to the proof of Theorem 1.3.1. $\square$

Theorem 4.3.2 *Let $\mathcal{S}$ be a near decagon of order $(1,t;\mu,1,1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that, for $\epsilon=-1,1$, the number $2+2\epsilon\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ is compatible with the order of θ. If $g_i, i=0,1,2$ is the number of points for which $\mathrm{d}(x,x^\theta)=2i$ then for some integers k_1, k_2, k_3 and k_4 there holds*

$$
\begin{aligned}
&k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+\\
&k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+\\
&k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+\\
&k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+(2(1+t))^2=\\
&(t+1)(t+2)g_0+2(t+1)g_1+g_2.
\end{aligned}
$$

Proof. Since the entry on the i^{th} row and i^{th} column of QM is the number of lines incident with x_i and x_i^θ, we have $\mathrm{tr}(QM)=(t+1)g_0+g_1$. Hence

$$
\begin{aligned}
& M=A+(1+t)I\\
\Rightarrow\quad & QM=QA+(1+t)Q\\
\Rightarrow\quad & \mathrm{tr}(QM)=\mathrm{tr}(QA)+(1+t)\mathrm{tr}(Q)\\
\Rightarrow\quad & (1+t)g_0+g_1=\mathrm{tr}(QA)+(1+t)g_0\\
\Rightarrow\quad & \mathrm{tr}(QA)=g_1.
\end{aligned}
$$

The matrix $A^2=(a_{ij})$ is the matrix with $(t+1)$ along the main diagonal and on the other entries we have $a_{ij}=1$ if $\mathrm{d}(x_i,x_j)=4$ and $a_{ij}=0$ otherwise. Hence $\mathrm{tr}(QA^2)=(1+t)g_0+g_2$. It follows that

$$
\begin{aligned}
& \mathrm{tr}(QM^2)\\
=\ & \mathrm{tr}(Q(A+(1+t)I)^2)\\
=\ & \mathrm{tr}(QA^2)+2(1+t)\mathrm{tr}(QA)+(1+t)^2\mathrm{tr}(Q)\\
=\ & (1+t)g_0+g_2+2(1+t)g_1+(1+t)^2g_0\\
=\ & (2+t)(1+t)g_0+2(1+t)g_1+g_2.
\end{aligned}
$$

The rest of the proof is completely similar to the proof of Theorem 1.4.4. □

Theorem 4.3.3 *Let $\mathcal{S}$ be a near decagon of order $(1,t;\mu,1,1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that, for $\epsilon=-1,1$, the number $2+2\epsilon\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ is compatible with the order of θ. If $g_i, i=0,1,2,3$ is the number of points for which $\mathrm{d}(x,x^\theta)=2i$ then for some integers k_1, k_2, k_3 and k_4 there holds*

$$
\begin{aligned}
&k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^3+\\
&k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^3+
\end{aligned}
$$

$$
\begin{aligned}
&k_3((t+1) + \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^3 + \\
&k_4((t+1) - \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}^{\,3}) + (2(1+t))^3 = \\
&(t+1)^2(t+4)g_0 + (3t^2 + 8t + 4)g_1 + 3(t+1)g_2 + g_3.
\end{aligned}
$$

Proof. From the proof of Theorem 4.3.2 it follows that $\mathrm{tr}(QA) = g_1$ and $\mathrm{tr}(QA^2) = (1+t)g_0 + g_2$. The matrix $A^3 = (a_{ij})$ is the matrix with $a_{ij} = 2t+1$ if $\mathrm{d}(x_i, x_j) = 2$, $a_{ij} = 1$ if $\mathrm{d}(x_i, x_j) = 6$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^3) = (2t+1)g_1 + g_3$. It follows that

$$
\begin{aligned}
& \mathrm{tr}(QM^3) \\
= \; & \mathrm{tr}(Q(A + (1+t)I)^3) \\
= \; & \mathrm{tr}(QA^3) + 3(1+t)\mathrm{tr}(QA^2) + 3(1+t)^2\mathrm{tr}(QA) + (1+t)^3\mathrm{tr}(Q) \\
= \; & (2t+1)g_1 + g_3 + 3(1+t)((1+t)g_0 + g_2) + 3(1+t)^2 g_1 + (1+t)^3 g_0 \\
= \; & (1+t)^2(4+t)g_0 + (4 + 8t + 3t^2)g_1 + 3(1+t)g_2 + g_3.
\end{aligned}
$$

The rest of the proof is completely similar to the proof of Theorem 1.5.5 □

Theorem 4.3.4 *Let $\mathcal{S}$ be a near decagon of order $(1, t; \mu, 1, 1)$ and let θ be an automorphism of $\mathcal{S}$. Assume that, for $\epsilon = -1, 1$, the number $2 + 2\epsilon\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu$ is compatible with the order of θ. If $g_i, i = 0, 1, 2, 3, 4$ is the number of points for which $\mathrm{d}(x, x^\theta) = 2i$ then for some integers k_1, k_2, k_3 and k_4 there holds*

$$
\begin{aligned}
&k_1((t+1) + \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^4 + \\
&k_2((t+1) - \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^4 + \\
&k_3((t+1) + \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^4 + \\
&k_4((t+1) - \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^4 + (2(1+t))^4 = \\
&(t+1)((t+1)^2(t+7) + 2t + 1)g_0 + 4(t+1)((2t+1) + (t+1)^2)g_1 + \\
&((3t+1) + 6(t+1)^2)g_2 + 4(t+1)g_3 + \mu g_4.
\end{aligned}
$$

Proof. From the proofs of Theorems 4.3.2 and 4.3.3 it follows that $\mathrm{tr}(QA) = g_1$, $\mathrm{tr}(QA^2) = (1+t)g_0 + g_2$ and $\mathrm{tr}(QA^3) = (2t+1)g_1 + g_3$. The matrix $A^4 = (a_{ij})$ is the matrix with $(t+1)(2t+1)$ along the main diagonal and on the other entries we have $a_{ij} = 3t+1$ if $\mathrm{d}(x_i, x_j) = 4$, $a_{ij} = \mu$ if $\mathrm{d}(x_i, x_j) = 8$ and $a_{ij} = 0$ otherwise. Hence $\mathrm{tr}(QA^4) = (t+1)(2t+1)g_0 + (3t+1)g_2 + \mu g_4$. It follows that

$$
\begin{aligned}
& \mathrm{tr}(QM^4) \\
= \;& \mathrm{tr}(Q(A+(1+t)I)^4) \\
= \;& \mathrm{tr}(QA^4)+4(1+t)\mathrm{tr}(QA^3)+6(1+t)^2\mathrm{tr}(QA^2)+4(1+t)^3\mathrm{tr}(QA)+(1+t)^4\mathrm{tr}(Q) \\
= \;& (t+1)(2t+1)g_0+(3t+1)g_2+\mu g_4+4(1+t)((2t+1)g_1+g_3)+ \\
& 6(1+t)^2((1+t)g_0+g_2)+4(1+t)^3g_1+(1+t)^4g_0 \\
= \;& (t+1)((t+1)^2(t+7)+2t+1)g_0+4(t+1)((2t+1)+(t+1)^2)g_1+ \\
& ((3t+1)+6(t+1)^2)g_2+4(t+1)g_3+\mu g_4.
\end{aligned}
$$

The rest of the proof is totally analogous to the proof of Theorem 1.6.4 $\square$

Note that, by Lemma 1.2.5, the integers k_1, k_2, k_3 and k_4 in Theorems 4.3.1, 4.3.2, 4.3.3 and 4.3.4 are the same.

Suppose now that we have a duality in the underlying partial quadrangle, then we know that $g_0 = 0$, $g_2 = 0$ and $g_4 = 0$. Because of Theorems 4.3.1, 4.3.2, 4.3.3 and 4.3.4, we have the following equations:

$$
\begin{aligned}
& k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+ \\
& \quad k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+ \\
& \quad k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+ \\
& \quad k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})+2(1+t)=g_1,
\end{aligned}
$$

$$
\begin{aligned}
& k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+ \\
& \quad k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+ \\
& \quad k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+ \\
& \quad k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^2+(2(1+t))^2= \\
& \quad 2(t+1)g_1,
\end{aligned}
$$

$$
\begin{aligned}
& k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^3+ \\
& \quad k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^3+ \\
& \quad k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^3+ \\
& \quad k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu}^{\,3})+(2(1+t))^3= \\
& \quad (3t^2+8t+4)g_1+g_3,
\end{aligned}
$$

$$
\begin{aligned}
&k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+(2(1+t))^4=\\
&\quad 4(t+1)((2t+1)+(t+1)^2)g_1+4(t+1)g_3.
\end{aligned}
$$

Because g_0, g_2 and g_4 are 0, we know that $g_1+g_3+g_5=2(1+t+t^2+\frac{t^3+t^4}{\mu})$. This can also be taken into account when dealing with the above formulae.

Corollary 4.3.5 *Suppose that θ is a duality of a partial quadrangle of order (t,t,μ). If $2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ and $2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ are no squares, and if they both are compatible with the order of θ, then θ has $1+t$ absolute points and $1+t$ absolute lines and there are $(1+t)t^2$ points which are mapped to a line at distance 3 and $(1+t)t^2$ lines which are mapped to a point at distance 3. Hence there are $\frac{t^3(1-\mu)+t^4}{\mu}$ points which are mapped to a line at distance 5 and $\frac{t^3(1-\mu)+t^4}{\mu}$ lines which are mapped to a point at distance 5.*

Proof. If the mentioned quantities are indeed not perfect squares, then after some computation we deduce $k_1=k_2=k_3=k_4=0$ from the fact that the right hand sides of the above equalities are integers, and hence so are the left hand sides. The result follows, taking into account that, for a given duality, there are equally many points mapped onto lines at distance i as there are lines mapped onto points at distance i, for all i. □

Part II

Automorphisms of some Geometries with Restricted Displacements

Introduction

Brown & Abramenko [2] show that every automorphism of an irreducible non-spherical building has infinite displacement. Their method also gives information about the spherical case. For instance, in the rank 2 case, every automorphism maps some chamber to a chamber at codistance one, and if the diameter of the incidence graph is even (odd), then any duality (collineation) maps some chamber to an opposite one. For projective planes, this shows that collineations behave normally, where 'normal' means that at least one chamber is mapped onto an opposite one. However, it is easily seen that also dualities of projective planes behave normally. Counterexamples to this normal behaviour are given in [2], attributed to Van Maldeghem, and consist of symplectic polarities in projective spaces and central collineations in generalized polygons of even diameter. The goal of Part II is to classify some 'abnormal' automorphisms, which we will call 'domestic'.

The main result of Section 5 of [2], also proved earlier, using entirely different methods, by Leeb [32], asserts that every automorphism of any (thick) spherical building is not T-domestic, for some type subset T. Hence being not T-domestic seems to be the rule, and so it is worthwhile to look at automorphisms which are T-domestic, for some T. In this thesis we initiate such study, and start with a complete classification of all domestic automorphisms of generalized quadrangles, and all domestic automorphisms of projective spaces. These seem to be the first interesting cases allowing for a complete and rather explicit classification. We also show that generalized n-gons, for n odd, do not admit domestic automorphisms. Concerning polar spaces, we characterize the fixed point structure of i-domestic collineations, with i odd, and of collineations that are both i- and $(i+1)$-domestic, for i even.

These examples will enable us to draw some general conclusions. Indeed, it seems that J-domestic automorphisms imply a rather large fixed point set, and usually the simplexes of type J get not mapped any further then simplexes that have a point in common with their inverse image (and that common point is fixed). This also implies that the displacement of the J-simplexes is a lot smaller than opposition, and not just one or two units smaller.

A second conclusion is that in many cases J-domestic implies J'-domestic for a subset $J' \subseteq J$, with J' 'a lot smaller' than J. But this is not always the case. For generalized polygons, for instance, this gives rise to one of the most intriguing questions put forward by our research: the only known domestic automorphisms of generalized polygons that are neither point-domestic nor line-domestic all have order 4 and occur in the finite case with small parameters.

There is a second weird observation to be made: almost all exceptional cases that will emerge in Part II have their origin in symplectic polarities: (1) these are the only domestic dualities in projective spaces, (2) they provide the generalized polygons mentioned in the previous paragraph, either directly, or by Payne-derivation, or by being embedded in the related geometry (except for the generalized hexagon of order $(8, 2)$), and (3) the latter geometries (symplectic polar spaces) do not behave as nicely, with respect to Tits-diagrams and with respect to point-domestic automorphisms, as the other polar spaces do.

5

Projective spaces

In this chapter, we characterize symplectic polarities as the only dualities of projective spaces that map no chamber to an opposite one. This implies a complete characterization of all J-domestic dualities of an arbitrary projective space for all type subsets J. We also completely characterize and classify J-domestic collineations of projective spaces for all possible J. In particular this includes all domestic collineations.

5.1 Domestic dualities

In this section we will prove the following theorem.

Theorem 5.1.1 *Every domestic duality of a projective space is a symplectic polarity. In particular no even dimensional projective space admits domestic dualities.*

It is clear that in any one-dimensional projective space, the only domestic collineation is the identity, and for any domestic duality, all elements are absolute elements (and this can be considered as a symplectic polarity).

This is enough to get an induction started. Note that the problem only makes sense for finite dimensional projective spaces as infinite ones are never self dual.

We first prove some lemmas which are independent of the induction hypothesis.

Lemma 5.1.2 *Let θ be a duality of a projective space of dimension $d > 1$ with the property that every point is absolute. Then θ is a symplectic polarity.*

Proof. Suppose by way of contradiction that θ is not a polarity. Then there is some point x for which $x' = x^{\theta^2} \neq x$. Consequently also $H := x^\theta$ and $H' = x'^\theta$ are different, and we can choose a point $y \in H$, $y \notin H'$. Since lines are thick, there is a $z \in yx'$, $z \notin \{x', y\}$. Since $z \in H$, we have $z^\theta \ni H^\theta = x'$. Similarly, $x' \in y^\theta$. By assumption $z \in z^\theta$ and $y \in y^\theta$. Hence the line yz is in $z^\theta \cap y^\theta = (zy)^\theta \subseteq x'^\theta$. This contradicts $yz \notin H'$. Consequently θ is a polarity and hence a symplectic polarity as every point is an absolute one. □

Lemma 5.1.3 *If a line contains at least one non-absolute point, and $|\mathbb{K}| > 2$, then it contains at least two non-absolute points.*

Proof. Assume by way of contradiction that the line L contains exactly one non-absolute point x. Then x^θ intersects L in some point $y \neq x$. Since, by assumption, $y \in y^\theta$, we see that $L^\theta \cap L = \{y\}$. If $u_i \in L$, $i = 1, 2$, $x \neq u_i \neq y$, then $u_i^\theta = \langle u_i, L^\theta \rangle = \langle L, L^\theta \rangle$, implying $u_1 = u_2$, and so $|\mathbb{K}| = 2$. □

In view of the induction procedure, we assume that for given $d > 0$ the only domestic dualities of a projective space of dimension $d' < d$ are the symplectic polarities for odd d', and we assume that θ is a domestic duality of a d-dimensional projective space Π.

In view of Lemma 5.1.2, it suffices to show that every point of Π is absolute. Let, by way of contradiction, x be a point which is not absolute, and let H be its image under θ. For any subspace S in H, the image $\langle S, x \rangle^\theta =: S'$ is a subspace of H, and the correspondence $S \mapsto S'$ is clearly a duality θ_H of H. Since for a subspace S of H we have that $\langle S, x \rangle$ is opposite $\langle S, x \rangle^\theta$ if and only if S is opposite $\langle S, x \rangle^\theta$ in H, it follows easily that this duality is domestic (because, by the foregoing remark, if C is a chamber in H, then it is mapped onto an opposite chamber in H if and only if the chamber $\{\langle x, S \rangle : S \in C \cup \{\emptyset\}\}$ of Π is mapped onto an opposite one). By induction d is even and θ_H is a symplectic polarity. It follows that every point of H is absolute. Now let z be any point in H. By construction of θ_H, the image $(xz)^\theta$ is equal to z^{θ_H}, and $(xz)^{\theta^2}$ is equal to the span of $\langle x, z^{\theta_H} \rangle^\theta$ and $H^\theta =: x'$. Note that $x' \notin H$. Since θ_H is symplectic, z is an absolute point and we see that $(xz)^{\theta^2} = x'z$.

Let us first assume that $x' \neq x$. Let y be the intersection of xx' and H. Since y is absolute and $y \in H$, the image y^θ contains xx'. Put $S = H \cap y^\theta$. Then for any point $u \in xx'$, the image u^θ contains S, but it can only contain u if $u = y$ (indeed, if it contains u, and $u \neq y$, then it contains xx' and hence coincides with y^θ, implying $u = y$ after all). It follows that all points of xx' except for y are non-absolute. But this now implies that all points of u^θ, for $y \neq u \in xx'$, are absolute, replacing x by u in the previous arguments. Now we pick a line not in y^θ meeting y^θ in a pre-chosen point $v \neq y$. Lemma 5.1.3 implies that v is absolute, or $|\mathbb{K}| = 2$. So, by Lemma 5.1.2, we may assume that $|\mathbb{K}| = 2$. In this case, all points of $x^\theta \cup x'^\theta$ are absolute. Let z be any point not in $x^\theta \cup x'^\theta$. Then z belongs to $y^\theta \setminus H$. Suppose moreover that $z \notin \{x, x'\}$. The line xz meets H in a point u that belongs to y^{θ_H}. Hence $y \in u^{\theta_H} \subseteq u^\theta$. Since also x' belongs to u^θ, we see that the line xx' is contained in u^θ. It follows that, since this line is not contained in x^θ, it is neither contained in z^θ. Since z^θ does contain y, we now see that z^θ does not contain x. Since z^θ also contains u, it cannot contain z, and so it is not an absolute point. We have shown that all points outside $x^\theta \cup x'^\theta$ are non-absolute. But now, interchanging the roles of x and z (and noting that the next paragraph is independent of the current one), we infer that all points of z^θ are absolute, and they cannot all be contained in $x^\theta \cup x'^\theta$, the final contradiction of this case.

Now we assume that $x' = x$. As before, we deduce that no point $u \notin H$ is absolute (taking $u \neq x$, considering the line ux and noting that $(ux)^\theta$ contains $ux \cap H$). But then all points of u^θ are absolute, for $u \notin H$. For $u \neq x$ we obtain points outside H that are absolute, contradicting what we just deduced.

So we have shown that the symplectic polarities are the only domestic dualities in projective space. This proves Theorem 5.1.1. □

This has a few consequences. We assume that the type of an element of a projective space is its projective dimension as a projective subspace.

Corollary 5.1.4 *Let J be a subset of the set of types of an n-dimensional projective space, $n \geq 2$. If either J contains no even elements, or n is even, or the ground field (if defined) is nonabelian, then there is no J-domestic duality. In all other cases, symplectic dualities are the only J-domestic dualities.*

Proof. This follows from the fact that any symplectic polarity maps an even-dimensional subspace to a non-opposite subspace, and there exists a subspace of any odd dimension that is mapped onto an opposite subspace. These claims are easy to check and well known. Further, there do not exist symplectic polarities in even-dimensional projective space, and in projective spaces defined over proper skew fields. □

We can actually compute the displacement of a symplectic polarity. To do this, we first remark that , if U is a subspace of even dimension, then U^ρ meets U in at least one point (otherwise the permutation of the set of subspaces of U sending a subspace W to $W^\rho \cap U$ would be a symplectic polarity, contradicting the fact that U has even dimension). Hence, if the projective space is $(2n-1)$-dimensional, the image of any chamber contains at least n elements that are not opposite their image. In order to "walk" to an opposite chamber, we need at least n steps. This shows that the codistance from a chamber to its image is at least n. We now show that this minimum is reached. Therefor, we consider the symplectic polarity ρ of $\mathsf{PG}(2n-1,\mathbb{K})$, with $\mathbb{K}$ a field, given by the standard alternating bilinear form

$$\sum_{i=1}^{n} X_{2i-1}Y_{2i} - X_{2i}Y_{2i-1},$$

where we introduced coordinates $(x_1, x_2, \ldots, x_{2n})$. Now we just consider the chamber C whose element of type i is given by the span of the first $i+1$ basis points (or, in other words, the set of points whose last $2n-i-1$ coordinates are zero). In dual coordinates, a straightforward computation shows that the element of type i of the image under ρ of C is given by putting the first $i+1$ coordinates equal to zero, if i is odd, and by putting the first $i-1$ coordinate equal to zero, together with the $(i+1)^{\text{st}}$ coordinate equal to zero, if i is even. Subsequently applying the coordinate change switching the $(2i-1)^{\text{st}}$ and $2i^{\text{th}}$ coordinates, for i taking the (subsequent) values $1, 2, \ldots, n$, we obtain a gallery of chambers ending in a chamber opposite C. This shows that minimal gallery codistance between a chamber and its image under a symplectic polarity in $(2n-1)$-dimensional space is equal to n.

5.2 J-domestic collineations

Now we consider collineations of projective spaces. Let us fix the projective space $\mathsf{PG}(n,\mathbb{K})$, with $\mathbb{K}$ any skew field. Let J be a subset of the type set. Define J to be *symmetric* if, whenever $i \in J$, then $n-i-1 \in J$. Then clearly, if J is not symmetric, then every collineation is J-domestic. Indeed, no flag of type J is in that case opposite any flag of type J. Hence, from now on, we assume that J is symmetric. We first prove two reduction lemmas. The first one reduces the question to type subsets of size 2, the second one reduces the question to single subspaces instead of pairs.

Lemma 5.2.1 *Let J be a symmetric subset of types for $\mathsf{PG}(n,\mathbb{K})$. Let i be the largest element of J satisfying $2i < n$. Then a collineation θ of $\mathsf{PG}(n,\mathbb{K})$ is J-domestic if and* *only if it is $\{i, n-i-1\}$-domestic.*

Proof. Clearly, if θ is $\{i, n-i-1\}$-domestic, then it is J-domestic. So assume that θ is J-domestic. Let i be as in the statement of the lemma. Suppose that θ is not $\{i, n-i-1\}$-domestic and let U and V be subspaces of dimension i, $n-i-1$, respectively, such that $U \subseteq V$ with $\{U, V\}$ opposite $\{U^\theta, V^\theta\}$, i.e., $U \cap V^\theta = V \cap U^\theta = \emptyset$.

Now choose in U any flag $\mathfrak{F}_{<i}$ of type $J_{<i}$, where with obvious notation, $J_{<i} = \{j \in J : j < i\}$. Let $\mathfrak{F}$ be an arbitrary extension of type J of the flag $\mathfrak{F}_{<i} \cup \{U, V\}$. Then $\mathfrak{F}$ is opposite $\mathfrak{F}^\theta$ if and only if each subspace $W \in \mathfrak{F}$ of type $j > n-i-1$ is disjoint from the unique subspace W' of $\mathfrak{F}^\theta_{<i}$ of type $n-j-1$ and each subspace $Z \in \mathfrak{F}_{<i}$ of type $j < i$ is disjoint from the unique subspace Z' of $\mathfrak{F}^\theta$ of type $n-j-1$. The latter is equivalent with saying that each subspace Y of $\mathfrak{F}$ of type $n-j-1 > n-i-1$ is disjoint from the unique subspace $Y' \in \mathfrak{F}^{\theta^{-1}}_{<i}$ of type j. So, we deduce that $\mathfrak{F}$ is opposite $\mathfrak{F}^\theta$ if, and only if, the flag $\mathfrak{F}_{>n-i-1}$ (with obvious notation) is opposite the two flags $\mathfrak{F}^\theta_{<i}$ and $\mathfrak{F}^{\theta^{-1}}_{<i}$. But one can always choose a flag opposite two given flags of the same type in any projective space. Indeed, this follows easily from the fact that we can always choose a subspace complementary to two given subspaces of the same dimension. Hence we have proved that θ is not J-domestic, a contradiction.

The lemma is proved. $\square$

So we have reduced the situation to symmetric type sets of two elements. With a similar technique, we reduce this further. But first a definition. For $i \leq n-i-1$ we say that a collineation is *i-$*$-domestic*, if θ maps no subspace of dimension i to a disjoint subspace.

Then we have:

Lemma 5.2.2 *Let $i \leq n-i-1$. Then a collineation θ of $\mathsf{PG}(n, \mathbb{K})$ is $\{i, n-i-1\}$-domestic if and only if it is i-$*$-domestic.*

Proof. It is clear that, if θ is i-$*$-domestic, then it is $\{i, n-i-1\}$-domestic. Suppose now that θ is $\{i, n-i-1\}$-domestic and not i-$*$-domestic.

Then there exists some subspace U of dimension i mapped onto a subspace U^θ disjoint from U. We can now choose a subspace V through U of dimension $n-i-1$ such that in V is disjoint from both U^θ and $U^{\theta^{-1}}$. That flag $\{U, V\}$ is mapped to an opposite flag, a contradiction.

The lemma is proved. $\square$

Note that we can dualize the previous definition and lemma. We will not do this explicitly, and we will not need to use this duality.

So, in order to classify all J-domestic collineations of $\mathsf{PG}(n,\mathbb{K})$ for arbitrary J, it sufficies to classify all i-$*$-domestic collineations, for all $i \leq n-i-1$. In order to do so, we may suppose that a given collineation θ is i-$*$-domestic, with $i \leq n-i-1$, but not j-$*$-domestic, for every $j < i$. We say that θ is *sharply i-$*$-domestic.*

In this setting, we can prove:

Theorem 5.2.3 *A collineation θ of $\mathsf{PG}(n,\mathbb{K})$ is sharply i-$*$-domestic, $i \leq n-i-1$, if and only if it pointwise fixes a subspace of dimension $n-i$, but it does not pointwise fix any subspace of larger dimension.*

Proof. First suppose that θ is sharply i-$*$-domestic. If θ pointwise fixed a subspace F of dimension $n-i+1$, then it would be $(i-1)$-$*$-domestic, since every subspace of dimension $i-1$ has at least one point in common with F and hence cannot be mapped onto a disjoint subspace.

We now show that θ fixes some subspace of dimension $n-i$ pointwise. To that aim, let U be a subspace of dimension $i-1$ which is mapped onto a disjoint subspace U^θ. Let V be an arbitrary i-dimensional subspace containing U and not contained in $X =: \langle U, U^\theta \rangle$. Since θ is i-$*$-domestic, the subspace V^θ has at least one point v in common with V. If $V \cap V^\theta$ contained a line, then that line would meet both U and U^θ and so both V and V^θ would be contained in X, a contradiction. It is now our aim to show that v is fixed. But we prove a slightly stronger statement.

Let W be any $(i+1)$-dimensional subspace of $\mathsf{PG}(d,\mathbb{K})$ containing V and intersecting V^θ in just v. Since $(i+1)+i \leq n$, such a subspace exists. If, on the one hand, W^θ met W in at least a plane, then such a plane would intersect V^θ in a line, contradicting our hypothesis $W \cap V^\theta = \{v\}$. If, on the other hand, $W \cap W^\theta$ were equal to $\{v\}$, then any i-dimensional subspace of W not through v and not through $v^{\theta^{-1}}$ would be mapped onto a disjoint subspace, contradicting i-$*$-domesticity. So $W \cap W^\theta$ is a line L (and note that L is of course not contained in V). We now claim that θ fixes L pointwise. Indeed, suppose that some point x on L is not fixed under θ. Then consider all subspaces of dimension i through x contained in W and not containing L. It is easy to see that all these subspaces have only the point x in common. Hence the images only have x^θ in common, and if $x \neq x^\theta$, then there is at least one image, say V'^θ, that does not contain x. But the intersection $V' \cap V'^\theta$ is contained in L. Since V' meets L in x and V'^θ does not contain x, it follows that V' and V'^{θ} are disjoint, contradicting i-$*$-domesticity. Our claim is proved.

Now let $\{W_i : i = 1, 2, \ldots, n-i\}$ be a set of $(i+1)$ dimensional subspaces containing V, not being contained in $\langle V, V^{\theta}\rangle$ and spanning $\mathsf{PG}(n, \mathbb{K})$. Such a set can easily be obtained by choosing a set of $n-i$ independent (and hence generating) points in the $(n-i-1)$-dimensional projective space $\mathrm{Res}(V)$ avoiding the subspace $\langle V, V^{\theta}\rangle$. Let $\{L_i : i = 1, 2, \ldots, n-i\}$ be the corresponding set of pointwise fixed lines ($L_i = W_i \cap W_i^{\theta}$). Since all L_i contain v and are pointwise fixed, θ pointwise fixes the space Z generated by the L_i, $i = 1, 2 \ldots, n-i$. The independency of the W_i in $\mathrm{Res}(V)$ now implies that the L_i are also independent in $\mathrm{Res}(v)$, and hence the subspace Z has dimension $n-i$, and that is what we had to prove.

Now suppose that θ pointwise fixes a subspace Z of dimension $n-i$, but it does not pointwise fix any subspace of larger dimension. Clearly, every subspace of dimension i meets Z and so is not mapped onto a disjoint subspace. Hence θ is i-$*$-domestic. But if it were j-$*$-domestic for $j < i$, then by the foregoing, it would pointwise fix a subspace of dimension $n-j > n-i$, a contradiction. Hence θ is sharply i-$*$-adjacent and the theorem is proved. $\square$

As a consequence, we can now characterize all domestic collineations of $\mathsf{PG}(n, \mathbb{K})$.

Corollary 5.2.4 *A collineation θ of an n-dimensonal projective space, $n \geq 2$, is domestic if and only if θ pointwise fixes a subspace of dimension at least $\frac{n+1}{2}$.*

Proof. For $n \geq 3$, this follows from Lemmas 5.2.1 and 5.2.2, and Theorem 5.2.3. For $n = 2$, every collineation is automatically point-domestic and line-domestic, so cannot be chamber-domestic (by Leeb [32]), unless it is the identity. $\square$

Concerning the maximal distance between a chamber and its image with respect to a domestic collineation, it is clear that this depends on the specific collineation. The maximum maximal distance occurs when the fixed point set is minimal, i.e., when the collineation is i-$*$-domestic, for $i \in \{\frac{n-1}{2}, \frac{n-2}{2}\}$. For n odd, the minimal codistance is in this case equal to 1, and for n even, it is equal to 3. In the other extreme, i.e., if the collineation fixes a hyperplane pointwise, then the maximal gallery distance between a chamber and its image is $2n+1$; this is codistance $\frac{n^2-3n-2}{2}$, which is rather large.

6 Small generalized polygons

In this chapter, we classify domestic collineations of some small generalized $2n$-gons, possibly under some rather ostensibly restrictive assumptions. The latter will happen for generalized quadrangles. The restrictions that we assume will be justified by arguments in Chapter 7. Indeed, it will turn out that these are the only domestic collineations that are neither point-domestic nor line-domestic (we call these *exceptional domestic collineations*). They all have order 4. For the other cases that we will treat, we only consider these out of curiosity, since we have no general result reducing the classification of domestic collineations to the point- and line-domestic ones, and to some additional well defined small generalized hexagons or octagons. Nevertheless, our results will show (1) that the classification of domestic generalized hexagons in the general case is not trivial, and (2) exceptional domestic collineations all seem to have order 4 and do exist for both quadrangles and hexagons. A general explanation of the phenomenon mentioned in (2) is not yet available.

The methods that we use are far from uniform. We usually just pick a convenient description of the polygon in question and start arguing. For the rather large cases, such as the split Cayley hexagons and the triality hexagon $\mathsf{T}(8,2)$, we have to rely on the character tables in the $\mathbb{ATLAS}$ [15] to count fixed points and fixed lines. But we also use the Benson-type formulae that we found in Chapter 1.

We begin in Section 6.1 with some small generalized quadrangles and then treat in Section 6.2 the split Cayley hexagons of order 2 en 3. Regarding the triality hexagon of order $(8,2)$, we only prove that there is an exceptional domestic collineation, but do not classificy them all. Note that hexagons of order $(2,2)$ and $(8,2)$ are classified (and they are isomorphic or dual to $\mathsf{H}(2)$, or isomorphic to $\mathsf{T}(8,2)$.

6.1 Small generalized quadrangles

Lemma 6.1.1 *Suppose* Γ *has order* $(2,2)$*, then up to conjugation there is exactly one domestic collineation which is neither point-domestic nor line-domestic and which fixes exactly one point and one line. It has order* 4*.*

Proof. We will use the following model of the generalized quadrangle of order $(2,2)$, see Chapter 6 of [38]. The points are pairs $\{i,j\}, i \neq j$ and $i,j \in \{1,2,3,4,5,6\}$. The lines are partitions $\{\{i,j\},\{k,l\},\{m,n\}\}$, where $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$. The incidence relation is given by inclusion. The full automorphism group is given by the natural action of the symmetric group S_6.

By assumption, we may assume that θ fixes the point $\{5,6\}$. It is easy to see that the only permutations in S_6 fixing the pair $\{5,6\}$ and not any other pair, are conjugate to the permutations $(1\ 2\ 3\ 4)$, $(1\ 2\ 3)(5\ 6)$ and $(1\ 2\ 3\ 4)(5\ 6)$. But the second permutation does not fix any line (contradicting our assumption); while the first one maps the flag $\{\{1,2\},\{\{1,2\},\{3,5\},\{4,6\}\}\}$ to the opposite flag $\{\{2,3\},\{\{2,3\},\{4,5\},\{1,6\}\}\}$. Also, one easily checks that $(1\ 2\ 3\ 4)(5\ 6)$ satisfies the given conditions. $\square$

Lemma 6.1.2 *Suppose* Γ *has order* $(2,4)$*, then up to conjugation there is exactly one domestic collineation which is neither point-domestic nor line-domestic and which fixes exactly one point and three lines. It has order* 4*.*

Proof. We will use the following model of the generalized quadrangle of order $(2,4)$, see Chapter 6 of [38]. The points are pairs $\{i,j\}$, with $i \neq j$ and $i,j \in \{1,2,3,4,5,6\}$, and the symbols $1,2,3,4,5,6$ and $1',2',3',4',5',6'$. The lines are triples $\{\{i,j\},\{k,l\},\{m,n\}\}$, where $\{i,j,k,l,m,n\} = \{1,2,3,4,5,6\}$, and triples $\{i,\{i,j\},j'\}$, where $i,j \in \{1,2,3,4,5,6\}$, $i \neq j$. Incidence is given by inclusion.

By assumption, we know that θ fixes exactly one point and three lines incident with that point. So we may assume that the point $\{5,6\}$ is fixed, and that also the lines $\{5,\{5,6\},6'\}$, $\{6,\{5,6\},5'\}$ and $\{\{5,6\},\{1,3\},\{2,4\}\}$ are fixed. Then θ interchanges the points 5 and $6'$ and the points 6 and $5'$. We now claim that the size of the orbit of $\{1,2\}$ under θ is 4. If not, then θ^2 fixes all points of $\{5,6\}^\perp$. Take a point $x \notin \{5,6\}^\perp$. Then x^{θ^2} is collinear with all points of $\{\{5,6\},x\}^\perp$, hence $x = x^{\theta^2}$ and hence $\theta^2 = 1$. Now take a line L on $\{1,2\}$ which is not incident with $\{5,6\}$. The image L^θ of L is incident with the point $\{1,2\}^\theta$ which is opposite $\{1,2\}$, hence, because θ is domestic, the lines L and L^θ are concurrent. If y is the intersection of L and L^θ, then it follows that $y^\theta = y$, a contradiction. Consequently we may assume, possibly by substituting θ with its inverse, that

$$\theta : \{1,2\} \mapsto \{2,3\} \mapsto \{3,4\} \mapsto \{1,4\} \mapsto \{1,2\}.$$

But now θ is completely determined, as every point x opposite $\{5,6\}$ is itself determined by the trace $\{\{5,6\},x\}^\perp$. For instance, the point 1 is collinear with $\{6',5',\{1,3\},\{1,2\},\{1,4\}\}$ which is mapped to $\{5,6,\{2,4\},\{2,3\},\{1,2\}\}$, and this set equals $\{\{5,6\},2'\}^\perp$. One obtains that θ is naturally induced by the permutation $\varphi := (1\ 2\ 3\ 4)(5\ 6)$ (in the sense that the image of the point $\{a,b\}$, $a \neq b$, under θ is the point $\{a^\varphi, b^\varphi\}$, and the images of the points a and b' under θ are $(a^\varphi)'$ and b^φ, respectively, for all $a,b \in \{1,2,\ldots,6\}$), and observe that the collineation induced by the permutation $(2\ 4)$ conjugates θ into its inverse. One easily checks that θ satisfies the conditions, which completes the proof of the lemma. $\square$

The following lemma belongs to folklore, but we provide a short proof for the sake of completeness.

Lemma 6.1.3 *Let $\mathcal{O}$ be a hyperoval in $\mathsf{PG}(2,4)$. Let θ be a collineation of $\mathsf{PG}(2,4)$ preserving $\mathcal{O}$. Then the companion field automorphism of θ is trivial if and only if the permutation induced on $\mathcal{O}$ by θ is even.*

Proof. It is well known that the stabilizer of $\mathcal{O}$ in $\mathsf{P\Gamma L}_3(4)$ is isomorphic to the symmetric group S_6, with natural action on $\mathcal{O}$. Since S_6 has a unique subgroup of index 2 — which is the alternating group A_6 — and since the intersecion of S_6 with the subgroup $\mathsf{PGL}_3(4)$ of index 2 of $\mathsf{P\Gamma L}_3(4)$ is a subgroup of index at most 2 in S_6, it suffices to prove that at least one element of S_6 does not belong to $\mathsf{PGL}_3(4)$. But this is easy: any collineation of $\mathsf{PG}(2,4)$ inducing a transposition on $\mathcal{O}$ fixes pointwise a quadrangle, and hence must be a Baer involution.

The lemma is proved. $\square$

Lemma 6.1.4 *Suppose Γ has order $(3,5)$, then up to conjugation there is exactly one domestic collineation which is neither point-domestic nor line-domestic, which has no fixed elements and for which exactly 48 points are mapped onto collinear points. It has order 4.*

Proof. We use the following model of the generalized quadrangle of order $(3,5)$ see Chapter 6 of [38]. The points are the points of a 3-dimensional affine space $\mathsf{AG}(3,4)$ over $\mathsf{GF}(4)$. The lines are the lines of $\mathsf{AG}(3,4)$ which meet the plane at infinity π in a point of a fixed hyperoval $\mathcal{O}$. Introducing coordinates X_1, X_2, X_3, X_4 in the projective completion $\mathsf{PG}(3,4)$ of $\mathsf{AG}(3,4)$, we may assume that π has equation $X_4 = 0$, and that the coordinates of the points of the hyperoval $\mathcal{O}$ are $(1,0,0,0), (0,1,0,0), (0,0,1,0), (1,1,1,0), (1,\epsilon,\epsilon^2,0)$ and $(1,\epsilon^2,\epsilon,0)$, where $\mathsf{GF}(4) = \{0,1,\epsilon,\epsilon^2\}$.

Suppose θ satisfies the assumptions of the lemma. By [38], θ is induced by a collineation of $\mathsf{PG}(3,4)$ stabilizing $\mathcal{O}$. Hence θ induces a permutation of $\mathcal{O}$. If it fixes some point x of $\mathcal{O}$, then, since θ does not fix any line of Γ, and the lines through x are mutually opposite, θ must map every point to a collinear point, a contradiction.

Hence θ induces a fixpoint free permutation on $\mathcal{O}$. Since it also acts fixpoint freely on $\mathsf{AG}(3,4)$, the action on $\mathcal{O}$ must have order 2 or 4.

Suppose first that the action of θ on $\mathcal{O}$ has order 4. Then because of the foregoing lemma the companion field automorphism of θ is the identity (indeed, the action on $\mathcal{O}$ is an even permutation). So, up to conjugation, the matrix of θ has the form

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ a & b & c & d \end{pmatrix}.$$

If $d \neq 1$, then this matrix has an eigenvector belonging to the eigenvalue d, which gives rise to a fixed point in Γ. Hence $d = 1$. Let L be a line of Γ containing the point $(1,\epsilon,\epsilon^2,0)$ at infinity. Then this line contains at most one point x with the property that x^θ belongs to L. Since by assumption θ maps 48 points to collinear points, there are at most 32 points x with xx^θ a line of Γ through one of $(1,\epsilon,\epsilon^2)$ and $(1,\epsilon^2,\epsilon)$. Consequently, there is at least one point y with yy^θ a line of Γ through $(1,0,0,0)$. We may choose the coordinates of y equal to $(0,0,0,1)$, and those of y^θ $(1,0,0,1)$. Hence $a = d = 1$ and $b = c = 0$ in the above matrix. But now the flag $\{(1,0,0,\epsilon), M\}$, with M the line through $(1,0,0,\epsilon)$ and $(0,1,0,0)$, is mapped onto the opposite flag $\{(1,\epsilon^2,0,1), M^\theta\}$, with M^θ the line through $(1,\epsilon^2,0,1)$ and $(0,0,1,0)$.

Hence θ induces a fixpoint free involution on $\mathcal{O}$. It follows that the companion field automorphism is nontrivial this time, and we may assume that the matrix of θ has the form

$$\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ a & b & c & d \end{pmatrix}.$$

We may assume that the point $(0,0,0,1)$ is mapped onto a collinear point, for which we may take without loss of generality the coordinates $(1,0,0,1)$. Hence $a = d$ and $b = c = 0$. The cases $d = \epsilon$ and $d = \epsilon^2$ are equivalent by conjugating with the map that squares each coordinate of every point. But if $a = d = \epsilon$, then θ would map the flag

$$\{(\epsilon,0,0,1), \langle(\epsilon,0,0,1),(0,1,0,0)\rangle\}$$

onto the opposite flag

$$\{(1,\epsilon^2,\epsilon^2,\epsilon), \langle(1,\epsilon^2,\epsilon^2,\epsilon),(0,0,1,0)\rangle\},$$

a contradiction. Hence θ is given by

$$\theta : (x_1,x_2,x_3,x_4) \mapsto (x_1^2,x_2^2,x_3^2,x_4^2)\begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}.$$

This shows the uniqueness part of the lemma. To show existence, we need to show that θ as given above is domestic, since it is clearly not point-domestic nor line-domestic. Also, one can check with an elementary calculation that θ does not fix any point of Γ.

One sees that θ^2 fixes π pointwise, and it also fixes all lines through the point $(0,1,1,0)$. Hence, since θ fixes exactly three lines through $(0,1,1,0)$ in π, and also at least one line off π (for instance the line $\langle(0,1,1,0),(\epsilon,0,0,1)\rangle$), it fixes exactly seven lines through $(0,1,1,0)$, and these seven lines form a Baer subplane in the projection from $(0,1,1,0)$. It follows that every point x of $\mathsf{AG}(3,4)$ is contained in at least one plane β through $(0,1,1,0)$ and fixed under θ. Suppose that x does not lie on one of the two affine fixed lines of β. Let o_1 and o_2 be the two points of $\mathcal{O}$ in β. Consider the line xo_1. Then both $(xo_1)^\theta$ and $(xo_1)^{\theta^{-1}}$ contain o_2 and must meet xo_1 in one of the two points of xo_1 not lying on a fixed line, as otherwise this intersection point would be fixed under θ, a contradiction. But it is easy to see that both possibilities lead to x^θ being collinear to x in Γ, since xx^θ contains either o_1 or o_2. Hence if x^θ is opposite x, then x is incident with a fixed line K of $\mathsf{AG}(3,4)$ through $(0,1,1,0)$. But then, for an arbitrary point o in $\mathcal{O}$, the line xo intersects the line $x^\theta o^\theta$, since both are contained in the same plane $\langle x, o, (0,1,1,0)\rangle$. This shows the assertion. □

The assumption of mapping exactly 48 points onto collinear points will be motivated in Lemma 7.1.6.

6.2 Some small generalized hexagons

6.2.1 The split Cayley hexagon of order $(2,2)$

We start with the split Cayley hexagon $\mathsf{H}(2)$ of order 2. The following tables can be extracted from page 14 of the $\mathbb{ATLAS}$ [15] (Section $\mathsf{G}_2(2)'$). We list all collineations up to conjugation in the full automorphism group $\mathsf{G}_2(2)$ with $\mathbb{ATLAS}$-notation. From the character table we deduce the number of fixed points and fixed lines and we geometrically derive the fixed element structure. Usually this is an elementary exercise which we will not explicitly perform but can be done by the reader. In the last column we mention the domesticity of the given collineation and that is exactly what we aim to prove in this subsection.

This will show:

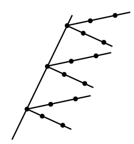
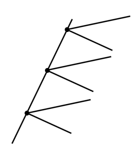
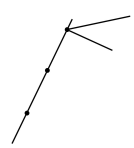
Theorem 6.2.1 *In the split Cayley hexagon of order* $(2,2)$*, there is, up to conjugation, exactly one domestic collineation which is neither point-domestic nor line-domestic. It has order* 4*.*

We will first prove the following general lemma, which is also true for any generalized n-gon, but we only need it for hexagons.

Lemma 6.2.2 *For every generalized hexagon* $\mathcal{H}$ *and every collineation* θ*, which fixes a point and every line incident with that point, or dually which fixes a line and every point incident with that line, there are no chambers which are mapped to a chamber at distance* 2.

Proof. By way of contradiction, we can assume that θ maps the flag $\{p, L\}$ to the flag $\{p', L'\}$ with $p\mathtt{I}L\mathtt{I}p'\mathtt{I}L'$ and $p \neq p', L \neq L'$. Suppose that we have a fixed line M and θ fixes also every point on that line. Take the projection $\mathrm{proj}_M p$ of p onto M, this is a fixed point. Suppose first $\mathrm{d}(p, M) = 5$, hence p, p^θ, $\mathrm{proj}_M p$ and the point collinear to both p and $\mathrm{proj}_M p$ are contained in a pentagon, unless p^θ is incident with $\mathrm{proj}_p M$, but then pp^θ is equal to L and hence L is fixed, a contradiction. If $\mathrm{d}(p, M) = 3$ we can make a similar reasoning. $\square$

We have divided the table into two parts according to whether the collineation belongs to the derived group or not. We first treat the collineations in the derived group $\mathsf{G}_2(2)'$.

	fixed points	fixed lines	structure of the fixed elements	conclusion
$2A$	15	7		axial collineation $\Rightarrow$ **point-domestic**
$3A$	0	9	a spread	**line-domestic**
$3B$	3	0	3 opposite points	not domestic
$4A$	3	7		not domestic
$4C$	3	3		**domestic** but neither point-domestic, nor line-domestic
$6A$	0	1	a line	not domestic
$7A$	0	0	no fixed elements	not domestic
$8A$	1	1	an incident point-line pair	not domestic
$12A$	0	1	a line	not domestic

In what follows θ is a collineation under consideration and we will denote the number of chambers which are mapped to a chamber at distance $i = 0, 1, \ldots, 6$ by f_i. Note that there are 189 chambers in total. Also we will apply Theorems 1.6.1, 1.6.2, 1.6.3, 1.6.4 and 1.6.5 to the dual of the double of $\mathsf{H}(2)$ without further notice (in this case $s = 2$ and $t = 1$); in particular the integers k_1, k_2, k_3, k_4 and k_5 are the ones mentioned in the statements of these theorems.

$\boxed{\textbf{Case } 2A}$

This is an axial collineation (one can easily see that points are always fixed or mapped to points at distance 4). Hence this collineation is point-domestic.

$\boxed{\textbf{Case } 3A}$

Every line, which is not fixed, is concurrent with a fixed line, hence it can only be mapped to a line at distance 4. Consequently it is line-domestic.

$\boxed{\textbf{Case } 3B}$

It is easy to see that $f_0 = 0$ and $f_1 = 9$. From Theorem 6.5.6 in [55] it follows that there are no points which are opposite all fixed points and hence, by a similar reasoning as in the proof of Lemma 6.2.2, it follows that $f_2 = 0$. Hence we obtain $f_3 = 18$, $f_4 = 36$, $f_5 = 72$ and $f_6 = 54$ (the formulae of Section 1.6 give $f_6 \equiv 54 \mod 96$ using the fact that $k_3 = k_4$ is an integer; but we also deduce that $f_4 = 63 - f_6/2$ and hence $f_6 = 54$, the rest follows easily). Hence because $f_6 \neq 0$, the collineation is not domestic.

$\boxed{\textbf{Case } 4A}$

Let L be the pointwise fixed line. Then clearly all points at distance 5 from L are mapped onto opposite points. If we assume that θ is domestic, then it follows that every line trough a point at distance 5 from L must be mapped onto a line at distance 4. This implies that θ is line-domestic. But this contradicts Theorem 7.2.1, which is proved independently. Hence the collineation is not domestic.

$\boxed{\textbf{Case } 4C}$

It is easy to see that $f_0 = 5$ and $f_1 = 8$. By Lemma 6.2.2 it follows that $f_2 = 0$. It follows that $f_6 = 0 \mod 96$ or hence $f_6 = 0$ or $f_6 = 96$. Suppose that $f_6 = 96$. Then we also calculate that $f_4 = 16$. Now applying Theorems 1.4.1 and 1.4.4 with 3 fixed points and 4 points mapped onto collinear points, we obtain after an elementary calculation that either 16 or 40 points are mapped onto points at distance 4, and either 40 or 16 to opposite points, respectively. But if only 16 go to opposite points, then at most 48 chambers can be mapped to opposite chambers, contradicting $f_6 = 96 > 48$. So there are precisely 16 points mapped to points at distance 4. Eight of these lie at distance 3 from the line L fixed pointwise, but none of those ones is contained in a chamber mapped to a chamber at distance 4, as is easily seen. Now let x be a point at distance 5 from L, with

$\mathrm{d}(x, x^\theta) = 4$, such that it is incident with a line M with $\mathrm{d}(M, M^\theta) = 4$, so that the flag $\{x, M\}$ is mapped onto a flag at distance 4. Then $\mathrm{d}(x, M^\theta) \in \{3, 5\}$. Both possibilities are equivalent (by considering θ^{-1}), and so we assume without loss of generality $\mathrm{d}(x, M^\theta) = 3$. Put $K = \mathrm{proj}_x M^\theta$. Then clearly $\mathrm{d}(K, K^\theta) = 4$. Similarly $\mathrm{d}(y, y^\theta) = 4$, with $y = \mathrm{proj}_{M^\theta} x$. Now letting y play the role of x, and going on like this, we obtain a cycle of length 16 (since the order of θ is 4) of points $x_0 \sim x_1 \sim \cdots \sim x_7 \sim x_0$ with $x_i^\theta = x_{i+2}$, subscripts taken modulo 8. Since θ^2 necessarily is an axial elation, and hence point-domestic, we deduce that $\mathrm{d}(x_i, x_{i+4}) = 4$. Let y_i, with i an integer mod 4, be the unique point collinear with both x_j and x_k, where $j \equiv k \equiv i \mod 4$, and $j \not\equiv k \mod 8$. Then $y_i^\theta = y_{i+2}$. It follows that θ^2 fixes each y_i. Since clearly y_i and y_{i+2} are not collinear (because the line $x_i y_i$ is opposite $x_{i+2} y_{i+2}$, with the subscripts modulo suitable integers), they must be at distance 4 from each other. Since there are three points per line, we automatically have that all points of the line $x_i y_i$, for every integer i, taken modulo 8 in x_i and modulo 4 in y_i, are mapped onto points at distance 4. But then we obtain at least 20 points mapped onto points at distance 4, contradicting the fact that we had 16.

Hence $f_6 = 0$. We also know already that either 16 or 40 points are mapped onto an opposite one, and the same thing is true for lines. Hence we conclude that θ is domestic, but it is neither point-domestic nor line-domestic.

Case $6A$

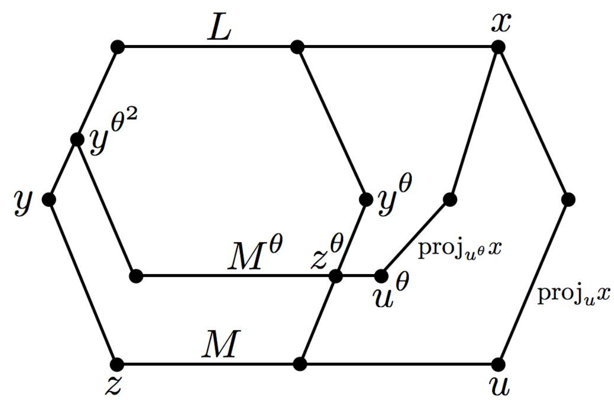
It is easy to see that $f_0 = 0$ and $f_1 = 3$. When we assume that $f_6 = 0$ we obtain $f_2 \equiv 72 \mod 96$ or hence $f_2 = 72$ or $f_2 = 168$. But in both cases it follows that f_4 is negative, a contradiction. Hence this collineation is not domestic.

Case $7A$

In this case $f_0 = 0$ and $f_1 = 0$. When we assume that $f_6 = 0$, we obtain $k_1 = -\frac{81}{32} + \frac{f_2}{32}$, hence $f_2 = 17 \mod 32$. We also obtain that $k_3 = -\frac{21}{32} + \frac{7}{96} f_2$, hence $f_2 = 9 \mod 96$, a contradiction. Hence this collineation is not domestic.

Case $8A$

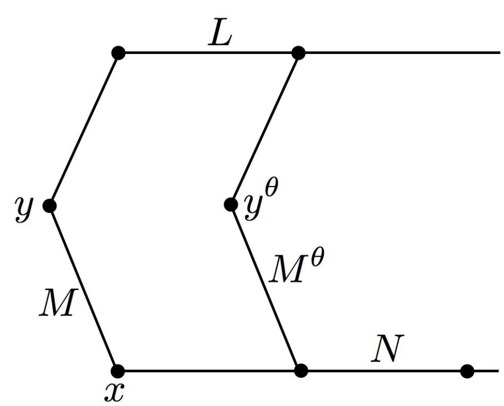
Suppose that θ is domestic. In this case we can not apply the theorems from Section 1.6 because the prime 2 is not compatible with the order 8 of θ (see Proposition 1.2.3). We remark that θ^2 belongs to the class $4A$, by the information provided by the $\mathbb{ATLAS}$. Let L be the fixed line and x the fixed point. Then it is easy to check that θ^2 fixes the line L pointwise, and hence also fixes all lines concurrent with L. Let z be a point opposite x. Let y be the unique point collinear with z and at distance 3 from L. Then the length of the orbit of y is 4. Since we assume that θ is domestic, and since y and y^θ are opposite, the line yz is mapped onto a line at distance 4. Let M be the unique line meeting yz and its image.



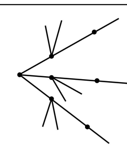
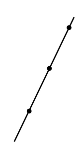
We may assume, by possibly renaming, that z is incident with M. It is easy to see that z^θ cannot be incident with M as z^θ lies at distance 4 from y^{θ^2} and the latter is collinear with y. Hence $\mathrm{d}(z, z^\theta) = 4$ and also $\mathrm{d}(M, M^\theta) = 4$. Since there are two collinear points on $(yz)^\theta$, with one on M and the other on M^θ, the points $u := \mathrm{proj}_M x$ and $u^\theta = \mathrm{proj}_{M^\theta} x$ are opposite. But the lines $\mathrm{proj}_u x$ and $\mathrm{proj}_{u^\theta} x$ are opposite, and so we have a chamber mapped to an opposite chamber. We conclude that θ is not domestic after all.

Case $12A$

Since the structure of fixed elements is the same, this is similar to Case 6A (by Proposition 1.2.4, the numbers 2 and 6 are compatible with 12). Alternatively we can also proof this in a geometric way as follows.



Assume that θ is domestic. Let L be the fixed line. We note that θ^2 also fixes only L, and that θ^3 fixes all lines concurrent with L (because it is in class $4A$). Take any point x at distance 5 from L. The line $M := \mathrm{proj}_x L$ is mapped onto a line at distance 4 (because $y := \mathrm{proj}_M L$ is mapped onto an opposite point). Hence there is a line N meeting both M and M^θ. We may take x on N. If x is mapped onto a point at distance 4, then we get a 24-gon consisting of points mapped onto points at distance 4 including x and $N \cap M^\theta$. All these 24 points are at distance 3 of one of the lines of the orbit of $K := \mathrm{proj}_y L$ (which has size 3). The "third" point on N is clearly mapped onto an opposite point. Hence for such a point, the previous assumption does not work, and we have to consider the case where x is mapped onto a collinear point. Then we obtain a 12-gon of points rotated by θ. The other point on M now is mapped onto an opposite point. So we have 12 points mapped to collinear ones, and 12 to opposites, and all 24 are at distance 3 of one of the lines in the orbit of K. Note that in this case the third point on the line N is mapped onto a point at distance 4, and hence we are back at our first case for x. We conclude that in any case we have exactly 15 points mapped onto collinear points, 24 points mapped onto points at distance 4 and 24 points mapped onto opposite points. But these numbers do not satisfy the congruences given by Theorems 1.4.1 and 1.4.4. Hence θ is not domestic anyway.

	fixed points	fixed lines	structure of the fixed elements	conclusion
$2B$	7	9		not domestic
$4D$	3	1		not domestic
$6B$	1	0	a point	not domestic
$8C$	1	1	an incident point-line pair	not domestic
$12C$	0	1	a line	not domestic

$\boxed{\textbf{Case } 2B}$

It is easy to see that $f_0 = 15$ and $f_1 = 18$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume that $f_6 = 0$, then it follows that $k_1 = 3/2$ and $k_3 = 7/2$, a contradiction. Hence this collineation is not domestic.

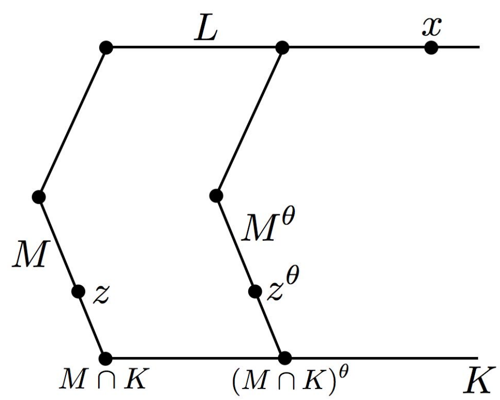
$\boxed{\textbf{Case } 4D}$

It is easy to see that $f_0 = 3$ and $f_1 = 6$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume that $f_6 = 0$, then it follows that $k_1 = -3/2$ and $k_3 = 1/2$, a contradiction. Hence this collineation is not domestic.

$\boxed{\textbf{Case } 6B}$

This is the dual case of $6A$.

$\boxed{\textbf{Case } 8C}$



We assume that θ is domestic. Let L be the fixed line and x the fixed point. Let M be a line at distance 4 from L and at distance 5 from x. Since we assume that θ is domestic, and since $\mathrm{proj}_M x$ is mapped onto an opposite point, the line M must be mapped onto a line at distance 4 from M. Let K be the line meeting both M and M^θ. Similarly as in Case $8A$, the assumption that $K \cap M$ is mapped onto a point at distance 4 leads to the conclusion that θ is not domestic. Hence we may assume that the intersection point $K \cap M$ is mapped onto a collinear point. By the free choice of M, this implies that we have exactly 18 points mapped onto collinear points (including two points on L). Now, since every line at distance 3 from x and 4 from L is mapped onto an opposite line, and since we assume that θ is domestic, all 16 points at distance 4 from x on these lines are mapped onto points at distance 4. We apply Theorems 1.4.1 and 1.4.4 with 1 fixed point and 18 points mapped to collinear points and obtain that either 12 or 36 points are mapped onto points at distance 4, whereas either 32 or 8 points, respectively, are mapped to opposites. Since we already have 16 points mapped onto points at distance 4, we conclude that there are precisely 8 points mapped onto opposite points. All points at distance 3 from L and 4 from x qualify for this, and there are already 8 of them. But also the point z on M different from $K \cap M$ and from $\mathrm{proj}_M x$ is mapped onto an opposite, and this is a contradiction. We conclude that θ is domestic after all.

$\boxed{\textbf{Case } 12C}$

Here we again refer to the Case 6A, but we can also proof it in a geometric way as follows. Assume that θ is domestic. Just as in Case $12A$, we here have that a point at distance 5 from the fixed line L is either contained in a 24-gon of points mapped onto points at distance 4, or in a 12-gon of points mapped onto collinear points, or it is mapped to an opposite. Moreover, If we have a 12-gon as above, then we have 12 points mapped onto opposite points (the points at distance 5 from L lying on lines at distance 4 from L that contain a point of the 12-gon). But, again as in Case $12A$, we must have points of each kind, implying that we have exactly the same displacement numbers as in Case $12A$, leading to the same contradiction and the same conclusion.

6.2.2 The split Cayley hexagon of order $(3, 3)$

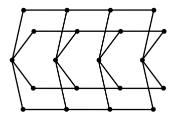
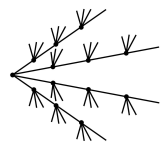
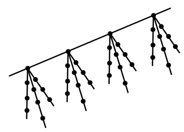
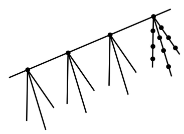
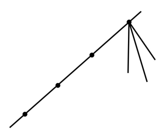
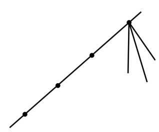
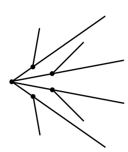
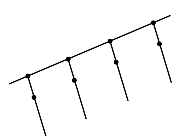
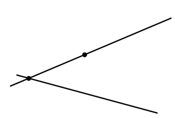
We will now go on with the generalized hexagon $\mathsf{H}(3)$ of order 3. Again we can extract the tables below from the $\mathbb{ATLAS}$ [15]. We shall use them to prove the following result.

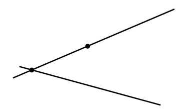
Theorem 6.2.3 *In the split Cayley hexagon of order* $(3, 3)$*, every collineation which is domestic is either point-domestic or line-domestic.*

Hence this generalized hexagon behaves much like the symplectic quadrangles of order ≥ 3.

The rest of this subsection is devoted to a proof of the above theorem. We use the same techniques as in the previous section.

In what follows θ is the collineation under consideration and we will denote the number of chambers which are mapped to a chamber at distance $i = 0, 1, \ldots, 6$ by f_i. Note that there are 1456 chambers in total. Also we will apply Theorems 1.6.1, 1.6.2, 1.6.3, 1.6.4 and 1.6.5 to the dual of the double of $\mathsf{H}(3)$ without further notice (in this case $s = 3$ and $t = 1$); in particular the integers k_1, k_2, k_3, k_4 and k_5 are the ones mentioned in the statement of these theorems.

	fixed points	fixed lines	structure of the fixed elements	conclusion
$2A$	20	20		not domestic
$3A$	13	40		central collineation $\Rightarrow$ **line-domestic**
$3B$	40	13		axial collineation $\Rightarrow$ **point-domestic**
$3C$	13	13		not domestic
$3D$	4	4		not domestic
$3E$	4	4		not domestic
$4A$	4	0	4 opposite points	not domestic
$4B$	0	4	4 opposite lines	not domestic
$6A$	5	8		not domestic
$6B$	8	5		not domestic
$6C$	2	2		not domestic

	fixed points	fixed lines	structure of the fixed elements	conclusion
$6D$	2	2		not domestic
$7A$	0	0	no fixed elements	not domestic
$8A$	2	0	2 opposite points	not domestic
$8B$	0	2	2 opposite lines	not domestic
$9A$	1	1	an incident point-line pair	not domestic
$9B$	1	1	an incident point-line pair	not domestic
$12A$	1	0	a point	not domestic
$12B$	0	1	a line	not domestic
$13A$	0	0	no fixed elements	not domestic

Case $2A$

It is easy to see that $f_0 = 48$ and $f_1 = 64$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume $f_6 = 0$ we obtain $k_3 = \frac{44}{7} + \frac{11}{3024} f_5$ and hence $f_5 \equiv 1296 \mod 3024$ or hence $f_5 = 1296$, but in this case we obtain $f_4 = -396$, a contradiction. This concludes that the collineation is not domestic.

Case $3A$

This is a central collineation, hence it is easy to see that this collineation is line-domestic.

Case $3B$

This is an axial collineation, hence it is easy to see that this collineation is point-domestic.

Case $3C$

It is easy to see that $f_0 = 25$ and $f_1 = 54$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume $f_6 = 0$ we obtain $k_3 = \frac{26}{7} + \frac{11}{3024} f_5$ and hence $f_5 = 1728 \mod 3024$, a contradiction. This concludes that the collineation is not domestic.

$\boxed{\textbf{Case } 3D}$

It is easy to see that $f_0 = 7$ and $f_1 = 18$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume $f_6 = 0$ we obtain $k_3 = -\frac{11}{14} + \frac{11}{3024} f_5$ and hence $f_5 \equiv 216 \mod 3024$. It follows that $f_5 = 216$, but then we obtain $f_3 = -513$, a contradiction. Hence this collineation is not domestic.

$\boxed{\textbf{Case } 3E}$

This case is totally analogous to the case $3D$.

$\boxed{\textbf{Case } 4A}$

It is easy to see that $f_0 = 0$ and $f_1 = 16$. Suppose that there exists a chamber $\{x, L\}$ which is mapped to a chamber at distance 2. Because θ has order 4 we obtain a cycle $x, L, x^\theta, L^\theta, x^{\theta^2}, L^{\theta^2}, x^{\theta^3}, L^{\theta^3}$ of length 8 (in the incidence graph) and hence we obtain a quadrangle, a contradiction. Hence $f_2 = 0$. When we assume $f_6 = 0$, we obtain $k_3 = -\frac{31}{21} + \frac{11}{3024} f_5$ and hence $f_5 = 2880 \mod 3024$, a contradiction. Hence this collineation is not domestic.

$\boxed{\textbf{Case } 4B}$

This is the dual case of $4A$.

$\boxed{\textbf{Case } 6A}$

It is easy to see that $f_0 = 12$ and $f_1 = 16$. By Lemma 6.2.2 it follows that $f_2 = 0$. When we assume $f_6 = 0$ we obtain $k_3 = -\frac{25}{42} + \frac{11}{3024} f_5$ and hence $f_5 \equiv 2088 \mod 3024$, a contradiction. This concludes that the collineation is not domestic.

$\boxed{\textbf{Case } 6B}$

This is the dual case of $6A$.

$\boxed{\textbf{Case } 6C}$

It is easy to see that $f_0 = 3$ and $f_1 = 10$. When we assume that $f_6 = 0$, we obtain $k_3 + k_4 + k_5 = \frac{55}{28} + \frac{13}{1008} f_5$ and hence $f_5 \equiv 468 \mod 1008$, or hence $f_5 = 468$, but in this case we obtain $18k_3 = -3/2 + f_2$, a contradiction, because k_3 and f_2 are integers. Hence this collineation is not domestic.

$\boxed{\textbf{Case } 6D}$

This case is totally analogous to the case $6C$.

$\boxed{\textbf{Case } 7A}$

It is easy to see that $f_0 = 0$ and $f_1 = 0$. When we assume that $f_6 = 0$, we obtain that $k_3 + k_4 + k_5 = \frac{13}{1008} f_5$, hence $f_5 \equiv 0 \mod 1008$ and hence, it follows that $f_5 = 0$ or $f_5 = 1008$. When we assume $f_5 = 0$, we obtain $k_3 - k_4 = \frac{104}{3}$, a contradiction. Hence $f_5 = 1008$, but in this case we obtain $k_3 - k_4 = -\frac{4}{3}$, again a contradiction. This concludes that $f_6 \neq 0$ and hence the collineation is not domestic.

$\boxed{\textbf{Case } 8A}$

It is easy to see that $f_0 = 0$ and $f_1 = 8$. When we assume $f_6 = 0$, we obtain that $k_3 + k_4 + k_5 = \frac{13}{14} + \frac{13}{1008} f_5$, hence $f_5 \equiv 936 \mod 1008$, so $f_5 = 936$. But in this case it follows that $k_3 - k_4 = 4/3$ a contradiction. This concludes that the collineation is not domestic.

$\boxed{\textbf{Case } 8B}$

This is the dual case of $8A$.

$\boxed{\textbf{Case } 9A}$

It is easy to see that $f_0 = 1$ and $f_1 = 6$. When we assume $f_6 = 0$, we obtain that $k_3 + k_4 + k_5 = \frac{27}{28} + \frac{13}{1008} f_5$, hence $f_5 \equiv 468 \mod 1008$, so $f_5 = 468$. But in this case it follows that $k_2 = -\frac{41}{4}$ a contradiction. We conclude that the collineation is not domestic.

$\boxed{\textbf{Case } 9B}$

This case is totally analogous to the case $9A$.

$\boxed{\textbf{Case } 12A}$

Here, clearly θ^3 belongs to class $4A$, while θ^6 belongs to class $2A$. Since multiples of θ centralize θ, the latter acts transitively on an orbit of 3 points x_1, x_2, x_3 opposite the unique fixed point x_0, in the position of the four points in the middle pictured in the table in the row of class $2A$. If we assume that θ is domestic, then the lines through x_i, $i = 1, 2, 3$ must be mapped onto lines at distance 4, and the only possibility is then that θ^3 fixes the four lines at distance 3 from all points x_i, $i = 0, 1, 2, 3$. But this contradicts the fixed point structure of an element of class $4A$, a contradiction. Hence θ cannot be domestic.

$\boxed{\textbf{Case } 12B}$

This is the dual case of $12A$.

$\boxed{\textbf{Case } 13A}$

This case is totally analogous to the case $7A$.

6.2.3 The triality hexagon of order $(8, 2)$

For the triality hexagon $\mathsf{T}(8, 2)$, we have no formulae relating the various displacements of the chambers. This makes a classification of exceptional domestic collineations rather difficult. However, a lot of collineations of $\mathsf{T}(8, 2)$ stabilize a subhexagon of order 2, and so we can rely on the results of Subsection 6.2.1. If the restriction of a collineation θ to such stabilized subhexagon of order 2 is not domestic, then θ itself is not domestic. This argument can be used for approximately half of the conjugacy classes of collineations of

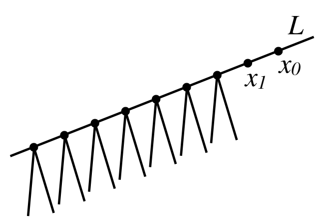
$\mathsf{T}(8,2)$. For the other collineations one needs additional techniques, such as coordinatization, or using some specific structural and geometric properties of the hexagons, such as regularity, intersection sets, transitivity, etc. At the moment of the writing of this thesis, we did not finish this job. The same holds for the smallest Ree octagon.

But two attractive classes of collineations are those that stabilize a subhexagon of order $(2,2)$ and induce in that subhexagon an exceptional domestic collineation. It becomes even more attractive to consider the class amongst these two that contains collineations of order 4. This is class $4A$ for $\mathsf{T}(8,2)$, with $\mathbb{ATLAS}$-notation. In view of the previous discussion, we content ourselves here with showing that class $4A$ contains exceptional domestic collineations.

We will use some terminology on elations not introduced here, but we refer the reader to [55].

Theorem 6.2.4 *The triality hexagon* $\mathsf{T}(8,2)$ *contains a conjugacy class of exceptional domestic collineations of order* 4. *Each such collineation stabilizes a subhexagon of order* 2 *in which the collineation induces an exceptional domestic collineation.*

Proof. Let θ be a collineation of class $4A$. It has the following fixed point stucture.



From the information in the $\mathbb{ATLAS}$, we deduce that the centralizer of θ is a group C of order $2^9 \cdot 7$. Also, looking at the various fixed point structures, one deduces that θ must be the extension to $\mathsf{T}(8,2)$ of a collineation θ^* of class $4C$ in $\mathsf{H}(2)$. The centralizer in $\mathsf{G}_2(2)'$ of θ^* has order 16. One can also see that the composition of two elations with perpendicular roots centralizes θ^* (and this lies outside $\mathsf{G}_2(2)'$). Hence, in $\mathsf{G}_2(2)$, the centralizer C^* of θ' has order 32. Let M be a line of $\mathsf{H}(2)$ opposite L, where L is the unique line of $\mathsf{H}(2)$ pointwise fixed under θ^*. Also, let x be the unique point on L fixed linewise under θ^*. Clearly C^* fixes the flag $\{x, L\}$ and hence is a subgroup of a Sylow 2-subgroup P^* of

$\mathsf{G}_2(2)$, which has order 64. Since the stabilizer of M in P^* has order 2 and consists of the identity together with a point-elation, and that point elation clearly does not belong to C^*, we conclude that C^*_M is trivial and so C^* acts transitively on the set of lines opposite L.

Now we have a look in $\mathsf{T}(8,2)$, and we embed the previous situation in $\mathsf{T}(8,2)$. The line L is now incident with 7 linewise fixed points, and two fixed points x_0, x_1 incident with only one fixed line, namely L (x, x_0 and x_1 are the three points of L in $\mathsf{H}(2)$). Let M again be opposite L. If some element of 2-power order fixes both L and M, then it must fix some point on L and hence interchange x_0 with x_1 and consequently have order 2. As above, such element can never centralize θ. Hence the orbit of M under C contains at least 2^9 elements, and so all elements opposite L. It also follows that the stabilizer C_M has order 7 and acts transitively on the seven points of L distinct from x_0, x_1. Anyway, since M is mapped onto a line at distance 4 from M, and C acts transitively on the lines opposite L, we see that all lines opposite L are mapped to lines at distance 4.

Hence the only lines of $\mathsf{T}(8,2)$ that are mapped onto opposite lines are the lines at distance 3 from x_0 or x_1 and 4 from L. Let K be such a line, and z a point on K at distance 5 from L. It is easy to check that there is a unique subhexagon H of order 2 containing x_0, x_1, z and K^θ (indeed, x_0, x_1 and z determine a unique subhexagon of order $(1,2)$, then K^θ intersects a line of that subhexagon, and so the claim follows). Since θ^2 is in the class $2A$, and hence is a central collineation, it stabilizes H, and so also θ stabilizes H. But in H, θ induces an exceptional domestic collineation (just by looking at its fixed point structure), and so z is mapped onto a point at distance 4. We conclude that θ is an exceptional domestic collineation. $\square$

7 Generalized polygons

In this chapter, we first classify the collineations of generalized quadrangles which map no chamber to an opposite one. Besides the three well-known exceptional cases occurring in the small quadrangles with orders $(2, 2)$, $(2, 4)$ and $(3, 5)$ that we treated in the previous chapter, all such collineations are either point-domestic or line-domestic. Up to duality, they fall into one of three classes: either they are central collineations, or they fix an ovoid, or they fix a large full subquadrangle. This settles the problem of domestic automorphisms in generalized quadrangles in the most satisfying way. We also give a complete answer for all generalized $(2n + 1)$-gons, namely, we prove that in such polygons no domestic automorphisms exist at all.

That leaves the case of generalized $2n$-gons, with $n \geq 3$. In this case, it follows from Leeb [32] that no duality is domestic, since the only nonempty type set invariant under duality is the full type set, and hence a flag mapped to an opposite must be a chamber. So it suffices to consider domestic collineations. But besides the small cases of the previous chapter, which show that the problem is nontrivial, we were not able to provide a complete classification as for the case $n = 2$. Hence this case is still open.

The chapter is structured as follows. In the first section, we deal with domestic generalized quadrangles and show that, besides the examples constructed in the previous chapter, every domestic collineation is a either point-domestic or line-domestic. We then classify the fixed point structures of such collineations. It will turn out that these structures are exactly the geometric hyperplanes or the dual geometric hyperplanes. In Section 7.2 we classify the fixed point structures of point-domestic and line-domestic collineations in generalized $2n$-gons, $n \geq 3$. We show that these are precisely the ovoidal or dual ovoidal subspaces, introduced by Brouns and Van Maldeghem for hexagons many years ago, see the introduction. So we classify ovoidal subspaces in generalized $2n$-gons. Finally, in Section 7.3, we show that there are no domestic dualities in generalized $(2n+1)$-gons. We treat the case $n = 2$ separately, as a warming up and exercise for the general case.

7.1 Domestic collineations of generalized quadrangles

In this section we prove the following theorem.

Theorem 7.1.1 *If θ is a domestic collineation of a (not necessarily finite) generalized quadrangle Γ of order (s,t) then we have one of the following possibilities.*

(i) *θ is either point-domestic or line-domestic.*

(ii) *$(s,t) \in \{(2,2),(2,4),(4,2)\}$, θ is neither point-domestic nor line-domestic and θ has fixed elements; a unique fixed chamber in case $(s,t) = (2,2)$, a unique point and three lines incident with it in case $(s,t) = (2,4)$, and the dual in the case $(s,t) = (4,2)$. Hence θ is as in Lemmas 6.1.1 and 6.1.2.*

(iii) *$(s,t) \in \{(3,5),(5,3)\}$, θ is neither point-domestic nor line-domestic, θ has no fixed elements and maps exactly 48 points to collinear points (for $s = 3$; for $t = 3$ the dual holds). Hence θ is as in Lemma 6.1.4.*

Also, if θ is line-domestic, then we have one of the following possibilities.

(i) *There are no fixed lines and the fixed points of θ form an ovoid.*

(ii) *There are fixed lines, but not two opposite ones. Then θ is a central collineation.*

(iii) *There are two opposite fixed lines and the fixed point-line structure is a full subquadrangle Γ' of Γ with the additional property that every line off Γ' meets Γ' in a unique point. In the finite case this is equivalent with Γ' having order $(s,t/s)$.*

And also, if θ is point domestic, then we have one of the following possibilities.

(*i*) *There are no fixed points and the fixed lines of θ form a spread.*

(*ii*) *There are fixed points, but not two opposite ones. Then θ is an axial collineation.*

(*iii*) *There are two opposite fixed points and the fixed point-line structure is an ideal subquadrangle Γ' of Γ with the additional property that every point off Γ' is incident with a unique line of Γ'. In the finite case this is equivalent with Γ' having order $(s/t, t)$.*

We shall prove this theorem in a sequence of lemmas and propositions. Throughout, let θ be a domestic collineation of a generalized quadrangle Γ with order (s,t). Our first big aim is to prove the following proposition.

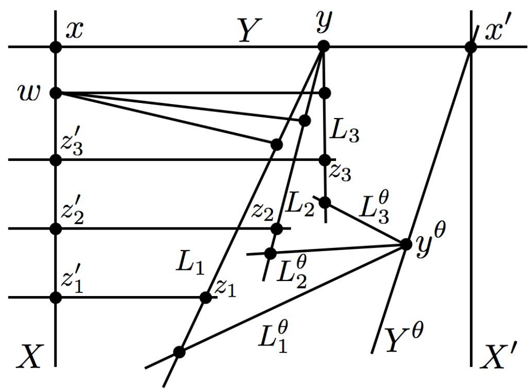
Proposition 7.1.2 *If $(s,t) \notin \{(2,2),(2,4),(4,2),(3,5),(5,3)\}$, then θ is either point-domestic or line-domestic. Also, if θ is neither point-domestic nor line-domestic and has fixed elements, then $(s,t) \in \{(2,2),(2,4),(4,2)\}$. If θ is neither point-domestic nor line-domestic and has no fixed elements, then $(s,t) \in \{(3,5),(5,3)\}$.*

We will prove this proposition in a few lemmas.

Lemma 7.1.3 *Suppose that $t \geq 3$, and that $s \geq 7$. If a line X is mapped onto an opposite line X', then $\{X, X'\}^{\perp}$ is fixed elementwise. Also, θ is point-domestic.*

Proof.

Suppose a line X is mapped onto an opposite line X' and some element Y concurrent with both X, X' is not fixed. We note that every point p on X is mapped onto $\mathrm{proj}_{X'} p$. Hence the intersection $x := X \cap Y$ is mapped onto $x' := X' \cap Y$, and so Y^θ is incident with x'. We assume, by way of contradiction, that $Y^\theta \neq Y$. Hence, every point on Y distinct from x and from x' is mapped onto an opposite point. Let y be such a point. Since θ is domestic, every line through y is mapped onto a concurrent line. Choose three such lines L_1, L_2, L_3, all distinct from Y. On L_i, $i \in \{1,2,3\}$, there is a unique point z_i that is mapped onto the intersection $L_i \cap L_i^\theta$ (and since $y^\theta \neq y$, we have $L_i \neq L_i^\theta$, so that the intersection is well defined). Let z_i' be the projection of z_i onto X, $i \in \{1,2,3\}$. Note that, applying θ, $\mathrm{proj}_{X'} z_i'$ is collinear to both $x^\theta = x'$ and z_i^θ. Since $s \geq 7$, there



exists a point w on X distinct from $x, z'_1, z'_2, z'_3, \mathrm{proj}_X z_1^\theta, \mathrm{proj}_X z_2^\theta, \mathrm{proj}_X z_3^\theta$. Let w_i be the projection of w onto L_i, $i \in \{1,2,3\}$. Then w_i^θ is opposite w_i. Hence the image of the line ww_i, $i = 1,2,3$, is concurrent with the line ww_i, contradicting the fact that at most two lines through w are mapped onto a concurrent one (namely, the line through w and w^θ, and the preimage of that line).

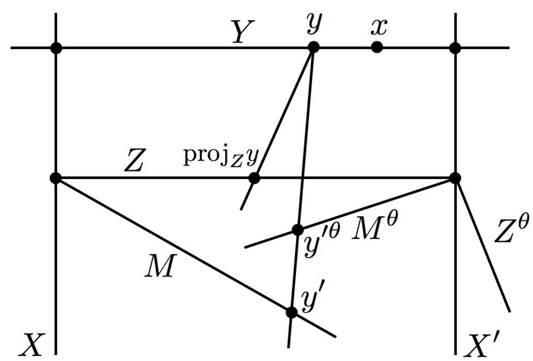
Suppose now that some point v is mapped onto an opposite point v^θ. Let v' be the projection of v onto X. Since v' is mapped onto a collinear but distinct point, and since vv' is not fixed, we see that vv' is mapped onto an opposite line, contradicting domesticity of θ. □

This lemma together with its dual proves Proposition 7.1.2 except for the cases $(s,t) \in \{(2,2),(2,4),(3,3),(3,5),(4,4),(4,6),(5,5),(6,6)\}$ (up to duality).

The following lemma makes some progress for the bigger values of s and t in the foregoing list, and shows that there are no fixed elements in the cases $(s,t) \in \{(3,5),(5,3)\}$.

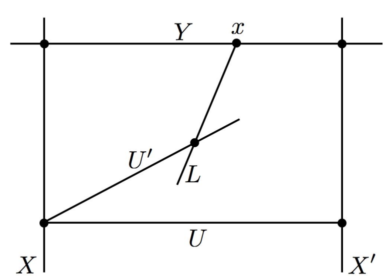
Lemma 7.1.4 *Suppose that Γ is finite, $s \geq 3$ and $t \geq 3$. If θ has fixed elements, then θ is either line-domestic or point-domestic.*

Proof. Suppose Γ is neither point-domestic nor line-domestic and suppose that θ fixes at least one point x. Then there exists a line X which is mapped onto an opposite line X'. Because the projection $x' := \mathrm{proj}_X x$ is collinear with its image x'^θ, it is easy to see that x is incident with the line $Y := x'x'^\theta$. Hence the line Y is fixed (which will allow us below to use the dual arguments of what follows). Suppose first that all lines concurrent with X and X' are fixed. We assumed that θ is not point-domestic, hence there exists a



point p which is mapped to an opposite point. This point can not lie on one of the fixed lines which are concurrent with X and X'. Consider the point $p' := \mathrm{proj}_X p$. The line pp' would be mapped to an opposite line and we obtain a flag $\{p, pp'\}$ which is mapped to an opposite flag, a contradiction. Hence we can assume that there is a line Z, concurrent with X and X' which is not fixed. Because the points on this line, not incident with X or X' are mapped to opposite points, it follows that every point y on Y, not incident with X or X', is fixed. Otherwise, the flag $\{\mathrm{proj}_Z y, \langle y, \mathrm{proj}_Z y\rangle\}$ would be mapped to an opposite flag.

There exists a line M through $X \cap Z$ different from X, Z and $Z^{\theta^{-1}}$. This line is mapped to an opposite line through $X' \cap Z$. Let y' be the projection of a point y of Y onto M. The image of y' is a point y'^θ on M^θ which is collinear to y'. Hence the line yy' is fixed. By equivalent reasons as before every point z on yy' different from y' and y'^θ is fixed. If z' is the point of X collinear with z, then $zz' \in \{X, X'\}^\perp$ is fixed. Hence exactly $s - 1 \geq 2$ lines of $\{X, X'\}^\perp$ are fixed under θ. Let U be one of these, $U \neq Y$.



If a line L through x is not fixed, then all points of L except for x are mapped onto opposites. Hence, since θ is domestic, the line U' through $X \cap U$ and concurrent with L must be mapped onto a concurrent line. Since U is fixed, U' must necessarily coincide with U. Hence U' is fixed, and so is L, a contradiction.

Because x was an arbitrary fixed point, it follows that every line through every fixed point is fixed. Hence we obtain a fixed subquadrangle of order $(s-2, t)$. Dually, this subquadrangle must also have order $(s, t-2)$, a contradiction. □

Remark 7.1.5 If we allow $s = 2$ and we assume that there exists a fixed element, then the first paragraph of the previous proof is still valid. Hence at least one line Z concurrent with both X and X' is not fixed. In fact, this immediately implies that also the remaining line concurrent with both X and X' is not fixed. This shows, using the first argument of the first paragraph of the previous proof, that x is the only fixed point. If $t = 2$, then clearly this argument can be dualized and we obtain a unique fixed chamber. Suppose now that $t = 4$. Then in the dual quadrangle we see that there is a line with exactly three fixed points. As there are no further fixed lines in this dual, θ fixes exactly one point and three concurrent lines in the quadrangle of order $(2, 4)$.

Lemma 7.1.6 *Suppose that* $(s,t) \in \{(2,2),(2,4),(3,3),(4,2),(4,4),(4,6),(5,5),(6,4),(6,6)\}$ *and that there are no fixed elements, then* θ *is the identity.*

Proof. If by way of contradiction θ is not the identity, then there is at least one element which is mapped to an opposite. So up to duality we may assume that there exists a line X which is mapped onto an opposite line X'. We will first count the number of points which are mapped to a collinear point. For every point p on X there are $t-1$ lines through p (including X) which are mapped to opposite lines. Hence all the points on these lines should be mapped to collinear points. On each of the two remaining lines through p, there are exactly two points which are mapped to collinear ones (twice including p). Hence $2(s-1)$ points collinear to p are mapped to opposite ones. Hence there are $(s+1)2(s-1)$ points which are mapped to opposite points and $(s+1)(t-2)s+3(s+1)$ points which are mapped to collinear points. Dually the number of lines mapped to opposite lines is equal to $2(t^2-1)$. We will now calculate this number in a different way.

We count the number of flags $\{y, Y\}$ which are mapped to a flag $\{y, Y\}^\theta$ where Y and Y^θ are opposite lines and y and y^θ are collinear points. Suppose that n is the number of lines which are mapped to an opposite line. Then there are $n(s+1)$ such flags. We can also count these flags as follows. There are $(s+1)(t-2)s+3(s+1)$ points which are mapped to a collinear point, through each of these points there are $t-1$ lines which are mapped to an opposite line. Hence $n(s+1)$ should be equal to $((s+1)(t-2)s+3(s+1))(t-1)$ and hence $n=((t-2)s+3)(t-1)$.

Combining the two previous paragraphs we obtain $3t+2s+st^2=3st+2t^2+1$. If $(s,t)\in\{(2,2),(2,4),(3,3),(4,2),(4,4),(4,6),(5,5),(6,4),(6,6)\}$, we obtain a contradiction. $\square$

The last identity of the previous proof is fulfilled for both $(s,t)=(3,5)$ and $(s,t)=(5,3)$. This is rather surprising, as the identity is far from being symmetric in s and t. Yet, the right exceptions emerge! Moreover, the previous proof shows that a domestic collineation in a quadrangle of order $(3,5)$ which is neither point- nor line domestic must have exactly $(s+1)(t-2)s+3(s+1)=48$ points mapped to a collinear point. This observation, together with Remark 7.1.5, completes the proof of the fact that domestic collineations of generalized quadrangles which are neither point-domestic nor line-domestic can only exist for orders $(2,2),(2,4)$ and $(3,5)$, up to duality. Moreover, in the first case, there is a unique fixed chamber, in the second case there is a unique fixed point and exactly three fixed lines, and in the last case there are no fixed elements and precisely 48 points mapped onto collinear ones.

Now we investigate what happens if θ is line-domestic.

Lemma 7.1.7 *If no line is mapped onto an opposite line, then we have one of the following possibilities.*

- *(i) There are no fixed lines and the fixed points of θ form an ovoid.*
- *(ii) There are fixed lines, but not two opposite ones. Then θ is a central collineation.*
- *(iii) There are two opposite fixed lines and the fixed point-line structure is a full subquadrangle Γ' of Γ with the additional property that every line off Γ' meets Γ' in a unique point. In the finite case this is equivalent with Γ' having order $(s,t/s)$.*

Proof. Suppose a line L is not fixed. Then there is a unique point x in the intersection of L and L^θ. If x were not fixed, then every line distinct from L and L^θ and incident with x would be mapped onto an opposite line, a contradiction. Hence x is fixed.

So, if no line is fixed, then every line is incident with a unique fixed point, and hence these form an ovoid.

Suppose now that there is at least one fixed line L and all fixed lines are concurrent and incident with some point z. Note that every point on any fixed line is a fixed point, as otherwise some line concurrent with that fixed line is mapped onto an opposite line. Suppose now that some line M concurrent with L is not fixed. Then every point x on M, not on L, is mapped onto an opposite point x^θ, and so every line through x is mapped onto a concurrent line. This implies the existence of a fixed point u off L, and hence of a fixed line L' different from L, but concurrent with it in the point z (by assumption). Note that M cannot be incident with z, as x and x^θ are both collinear with u, and u is collinear with z. In particular, it follows that all lines through z are fixed, and that we have a central collineation with center z.

If L and M are two opposite fixed lines, then, in view of the fact that every point on every fixed line is fixed, we see that the fixed point structure is a full subquadrangle Γ'. If some line off Γ' did not meet Γ', then it would be mapped onto an opposite line (as otherwise the intersection with its image is fixed and belongs to Γ' by definition of Γ'). In the finite case, it follows from [45] and [46] that Γ' has order $(s, t/s)$.

The lemma is proved. □

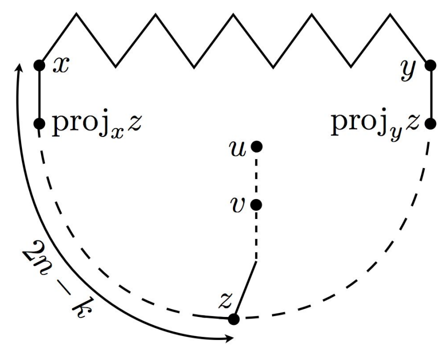
Combining the previous lemmas and their duals, we see that Theorem 7.1.1 is proved, if we also take into account Lemmas 6.1.1, 6.1.2 and 6.1.4.

7.2 Domestic collineations of generalized $2n$-gons

Let $\Gamma = (\mathcal{P}, \mathcal{L}, \mathtt{I})$ be a generalized $2n$-gon, with $n \geq 2$. We generalize the definition of an ovoidal subspace of a generalized quadrangle or hexagon to all generalized $2n$-gons. Therefor, we first define a *subspace* $\mathcal{S}$ of Γ as a subset of points and lines with the property that, as soon as two collinear points x, y belong to $\mathcal{S}$, then the line xy belongs to $\mathcal{S}$, and as soon as a line belongs to $\mathcal{S}$, then all points of that line belong to $\mathcal{S}$. Secondly, a subspace $\mathcal{S}$ is *ovoidal* if no element of Γ is at distance bigger than n from all points of $\mathcal{S}$, and if, whenever an element of Γ not in $\mathcal{S}$ is at distance less than n from some point of $\mathcal{S}$, then that element is at minimal distance from a unique point of $\mathcal{S}$. As mentioned in the introduction, these objects were introduced in [7] for generalized quadrangles and hexagons, and were subsequently classified. For generalized quadrangles, it is easy to see that an ovoidal subspace is just a *geometric hyperplane* (it is just a rephrasing of the definition of this object), and these come in three flavours: (1) ovoids, (2) point-perps and (3) full large subquadrangles (with a *large* subquadrangle, we mean a subquadrangle such that every line of Γ is incident with some point of the subquadrangle; in the finite case, if Γ has order (s, t), then any proper large full subquadrangle must have order $(s, t/s)$). For generalized hexagons, an ovoidal subspace is either (1) a distance-3 ovoid, or (2) the set of points and lines at distance at most 3 from a fixed line, or (3) a large full subhexagon, i.e., a full subhexagon with the property that every point of Γ is collinear to at least one point of the subhexagon (in the finite case, due to a result of Thas [48], this is equivalent with saying that, if Γ has order (s,t), then the subhexagon has order $(s, \sqrt{t/s})$).

In this subsection, we will generalize this classification to all $2n$-gons; the proof is a straightforward extension of the proof for hexagons. Then we will show:

Theorem 7.2.1 *The fixed element structure of a line-domestic collineation of any generalized $4n$-gon, $n \geq 1$, and the fixed element structure of a point-domestic collineation of any generalized $(4n+2)$-gon, $n \geq 1$, is an ovoidal subspace.*



So, let us first classify all ovoidal subspaces of a generalized $2n$-gon Γ. Let $\mathcal{S}$ be such a subspace. If $\mathcal{S}$ only contains mutually opposite points, then we obtain by definition a distance-n ovoid. Suppose now that $\mathcal{S}$ contains a pair of points x, y which are not opposite each other. Then the unique shortest path γ from x to y contains an element z at distance $< n$ from both x, y and hence, by definition, belongs to $\mathcal{S}$. Playing the same game with the pairs x, z and y, z, and so on, we easily see that all elements of γ belong to $\mathcal{S}$. We call this argument the "closing argument" for further reference. If $\mathcal{S}$ contains an apartment, then it is by definition a full subpolygon, and the defining properties of an ovoidal subspace are equivalent to the defining properties of a large subpolygon. There remains to consider the case that $\mathcal{S}$ contains points and at least one line L, but no apartment. Let x, y be two points of $\mathcal{S}$ at maximal distance $2k$. We claim that $k = n$. Indeed, if not, then consider any apartment Σ through x and y. Let z be the element of Σ in the middle of x and y contained in the longer path of Σ connecting x with y. Notice that neither $\mathrm{proj}_x z$ nor $\mathrm{proj}_y z$ belongs to $\mathcal{S}$ since this would violate the maximality of $\delta(x,y)$. Now, $z \notin \mathcal{S}$ and is at distance $> n$ from both x, y. Hence there is some element $u \in \mathcal{S}$ at distance $\leq n$ from z. Since $\mathrm{proj}_z u$ must be different from one of $\mathrm{proj}_z x, \mathrm{proj}_z y$, we see that there is a path of length $\leq 3n - k$ from u to either x or y containing z (a path does not contain twice the same element). Without loss of generality, we may assume that there is a path γ' from x to u of length $\ell \leq 3n - k$. Since $\mathrm{proj}_x z \notin \mathcal{S}$, we have $2n \leq \ell$. Let v be the point in γ' at distance $2n$ from x, measured in γ'. Then x and v are opposite in Γ and u is at most at distance $n - k$ from v. This implies using the triangle inequality that x and u are at least distance $n + k$ apart, contradicting the maximality of $2k < n + k$.

Hence the maximal distance between points of $\mathcal{S}$ is $2n$. Let x, y be opposite points of $\mathcal{S}$. Since $\mathrm{proj}_x L$ and $\mathrm{proj}_y L$ belong to $\mathcal{S}$, we see that at least one minimal path (of length $2n$) between x and y is entirely contained in $\mathcal{S}$. But then there is exactly one such path, say γ''. Let m be the middle element of γ''. We first claim that no element at distance $> n$ from m belongs to $\mathcal{S}$. Indeed, suppose $z \in \mathcal{S}$ and $\delta(z, m) = j > n$. By possibly considering the element at distance $n + 1$ from m contained in the shortest path between z and m, or $\mathrm{proj}_m x$ (the latter if $j = 2n$), we may assume that $j = n + 1$. By possibly interchanging the roles of x and y, we may also assume that $\mathrm{proj}_m z \neq \mathrm{proj}_m x$. Hence there is a unique apartment containing x, m, z and the "closing argument" now implies that all elements of this apartment belong to $\mathcal{S}$, contradicting the assumption that $\mathcal{S}$ does not contain an apartment.

Finally, we claim that every element z at distance $\leq n$ from m belongs to $\mathcal{S}$. Indeed, let $\delta(z, m)$ be equal to $j \leq n$ and suppose $z \notin \mathcal{S}$. First we treat the case $j < n$. Choose an element u at distance $n + j$ from m and n from z. By the foregoing, $u \notin \mathcal{S}$, and so, by definition, there is an element $x' \in \mathcal{S}$ at distance $\leq n$ from u. The sum of the lengths of the minimal paths between x' and m, between m and u, and between u and x' is strictly smaller than $4n$, hence the union of these paths is not a cycle but a tree with one branch point. In any case, since $\delta(u, x') < \delta(u, z)$, and $\delta(m, x') \geq \delta(m, z)$, we see that z must lie on the shortest path from m to x', contradicting the "closing argument". Now, if $j = n$, then z is a point, and by the foregoing, the line $\mathrm{proj}_z m$ belongs to $\mathcal{S}$, which implies by definition of ovoidal subspace that also z belongs to it.

Hence we have shown:

Theorem 7.2.2 *An ovoidal subspace of a generalized $2n$-gon is either a distance-n ovoid, or the set of points and lines at distance at most n from a fixed element (and that element is a point if n is even, and a line if n is odd), or a large full subpolygon.*

We will prove Theorem 7.2.1 for the $(4n+2)$-gons, in order to fix the ideas. The proof for

$4n$-gons is completely similar, and only requires renaming some points as lines and vice versa.

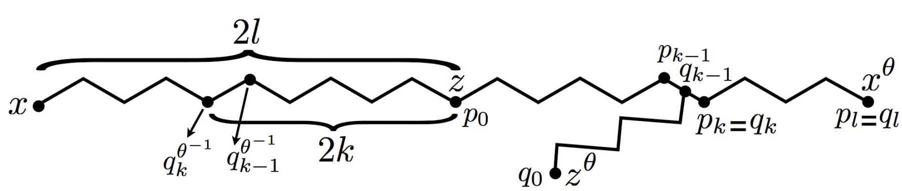
So let θ be a point-domestic collineation of a generalized $(4n+2)$-gon Γ, with $n \geq 1$.

Let L be a fixed line and suppose not all points on L are fixed. Let $x \mathtt{I} L$ be such that $x \neq x^\theta$. Complete the path (L, x) to a path γ of length $2n+1$ and let z be the last element of that path (observe that z is a point). Then the juxtaposition of the paths γ^{-1} and γ^θ yields a non-stammering path of length $4n+2$ connecting z with z^θ; hence z is opposite z^θ, contradicting point-domesticity. We have shown that every point on any fixed line is fixed itself. In particular, the fixed point structure is a subspace.

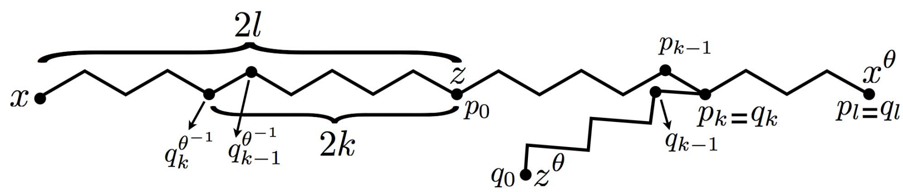
We now claim that for no point x it holds $\delta(x, x^\theta) = 4\ell + 2$, with $0 \leq \ell \leq n$. By assumption, this is true for $\ell = n$. Suppose $\ell < n$. Then, since Γ is thick, there exists a line M incident with x satisfying $M \neq \mathrm{proj}_x x^\theta$ and $M^\theta \neq \mathrm{proj}_{x^\theta} x$. It follows that $\delta(M, M^\theta) = 4\ell + 4$. Note that $4\ell + 4 \neq 4n + 2$, and so we can repeat this argument to obtain a point $y \mathtt{I} M$ with $\delta(y, y^\theta) = 4(\ell + 1) + 2$. Going on like this, we finally obtain a point opposite its image, contradicting the point-domesticity. Our claim is proved. We refer to this claim by $(*)$.

Now let x be any point of Γ. Since $\delta(x, x^\theta)$ is a multiple of 4, there is a unique point z with $\delta(x, z) = \delta(z, x^\theta) = \delta(x, x^\theta)/2$. We claim that $z = z^\theta$. Indeed, suppose by way of contradiction that $z \neq z^\theta$. Set $\delta(x, z) = 2\ell$. Consider the minimal paths $(p_0, p_1, \ldots, p_{\ell-1}, p_\ell)$ and $(q_0, q_1, \ldots, q_{\ell-1}, q_\ell)$ with $p_0 = z$, $q_0 = z^\theta$ and $p_\ell = x^\theta = q_\ell$. Let $k \leq \ell$ be minimal with the property that $p_k = q_k$. Note that k is well-defined since $p_\ell = q_\ell$, and note also that $k > 0$ as otherwise $z = z^\theta$. There are two possibilities.

(1) The point q_{k-1} is incident with the line $p_k p_{k-1}$. In this case, by minimality of k, $q_{k-1} \neq p_{k-1}$ and so $\delta(q_{k-1}^{\theta^{-1}}, q_{k-1}) = 4(k-1) + 2$, contradicting our claim (*).



(2) The line $q_k q_{k-1}$ is different from the line $p_k p_{k-1}$. Then $\delta(q_k^{\theta^{-1}}, q_k) = 4k$ and $\delta(q_{k-1}^{\theta^{-1}}, q_{k-1}) = 4(k-1) + 4 = 4k$, whereas $\delta(q_{k-1}^{\theta^{-1}}, q_k) = 4k - 2$ and $\delta(q_k^{\theta^{-1}}, q_{k-1}) = 4k + 2$.



Hence we easily see that for each point u on the line $(q_{k-1}q_k)^{\theta^{-1}}$ distinct from $q_{k-1}^{\theta^{-1}}$ and $q_k^{\theta^{-1}}$ we have $\delta(u, u^\theta) = 4k + 2$, again contradicting claim (*).

These contradictions prove our claim. Hence every point of Γ is at distance at most $2n$ from a fixed point. It also follows that every line is at distance at most $2n + 1$ from some fixed point.

There remains to show that, if some element x_0 is at distance $< 2n$ from some fixed point, then it is at minimal distance from a unique fixed point. Indeed, suppose x_0 is at minimal distance ℓ from both fixed points y and z. Let $(x_0, x_1, \ldots, x_{\ell-1}, x_\ell)$ be a minimal path from x_0 to $y = x_\ell$, and let $(x'_0, x'_1, \ldots, x'_{\ell-1}, x'_\ell)$, with $x'_0 = x_0$, be a minimal path from x_0 to $z = x'_\ell$. Let j be maximal with respect to the property $x_j = x'_j$. Then, since $2\ell < 4n + 2$, the element x_j belongs to the unique shortest path connecting z with y, and hence is fixed under θ. By minimality of ℓ, $j = \ell$ and consequently $y = x_\ell = x'_\ell = z$ after all.

The theorem is proved. □

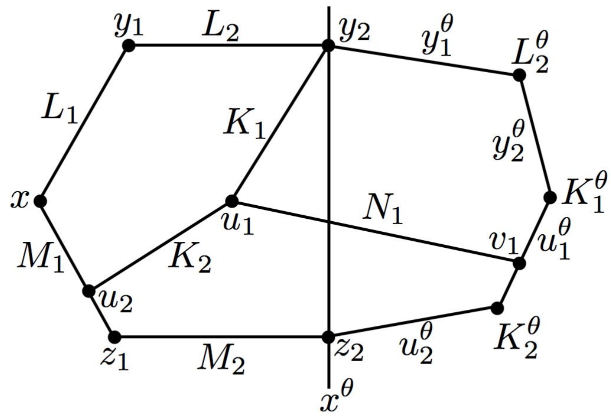
7.3 Domestic dualities of generalized $(2n + 1)$-gons

In this section we prove the nonexistence of domestic dualities in generalized $(2n+1)$-gons. It is convenient to first show this result for generalized pentagons, as this is an excellent exercise for the proof in the general case.

Theorem 7.3.1 *No duality of any generalized pentagon is domestic.*

Proof.

Let θ be a duality of a generalized pentagon Γ of order s, with $s > 1$ necessarily infinite. It is easy to see that there exists a point x which is mapped to an opposite line x^θ. We



will prove that there exists a chamber which is mapped to an opposite chamber. Assume by way of contradiction that there exists no chamber which is mapped to an opposite one. Take a path $xL_1y_1L_2y_2x^\theta$ and a path $xM_1z_1M_2z_2x^\theta$ from x to x^θ, with $L_1 \neq M_1$. The line L_1 is mapped to a point incident with x^θ, but because we assumed that there are no chambers which are mapped to an opposite one, the point L_1^θ is equal to the point y_2. For the same reason the line M_1 is mapped to the point z_2. Let u_2 be a point incident with the line M_1, with $u_2 \neq x$ and $u_2 \neq z_1$. Note that if x^{θ^2} is incident with M_1, also $u_2 \neq x^{\theta^2}$. Let $y_2K_1u_1K_2u_2$ be a path from the point y_2 to the point u_2. Then we also require that $K_1 \neq y_1^\theta$, which can be done easily. There exists a path of length 6 between the flags $\{K_1, u_1\}$ and $\{u_1^\theta, K_1^\theta\}$, namely the path

$$\{K_1, u_1\}, \{K_1, y_2\}, \{y_1^\theta, y_2\}, \{y_1^\theta, L_2^\theta\}, \{y_2^\theta, L_2^\theta\}, \{y_2^\theta, K_1^\theta\}, \{u_1^\theta, K_1^\theta\}.$$

Hence the flags $\{K_1, u_1\}$ and $\{u_1^\theta, K_1^\theta\}$ can not lie on a distance smaller than 4, otherwise we obtain a k-gon with $k < 5$. Because we assumed that the distance can not be equal to 5, the flags must lie at distance 4. Hence there exists a line N_1 which is incident with the point u_1 and which intersects u_1^θ in a point v_1 different from K_1^θ (otherwise the points u_1, y_2, L_2^θ and K_1^θ form a quadrangle) and different from the point K_2^θ (otherwise the points u_1, y_2, z_2 and K_2^θ form a quadrangle). For analogous reasons there exists a line N_2 which is incident with K_2^θ and which intersects K_2 in a point v_2 different from the point u_1 and the point u_2. Hence we obtain a quadrangle $u_1N_1v_1u_1^\theta K_2^\theta N_2v_2K_2$, a contradiction. $\square$

We generalize the previous theorem and its proof to all generalized $(2n+1)$-gons.

Theorem 7.3.2 *No duality of any generalized $(2n+1)$-gon is domestic.*

Proof. We may suppose that $n \geq 3$.

Let θ be a duality of a $(2n+1)$-gon Γ of order (s,t) with $s > 1$ and $t > 1$. It's easy to see that there exists a point x which is mapped to an opposite line x^θ. We will prove that there exists a chamber which is mapped to an opposite chamber. Assume by way of contradiction that no chamber is mapped to an opposite one. Take a path $xL_1y_1L_2y_2\cdots L_ny_nx^\theta$ and a path $xM_1z_1M_2z_2\cdots M_nz_nx^\theta$ from x to x^θ, $L_1 \neq M_1$. The line L_1 is mapped to a point incident with x^θ, but because we assumed that there are no chambers which are mapped to an opposite one, the point L_1^θ is equal to the point y_n. For the same reason the line M_1 is mapped to the point z_n. Let u_n be a point incident with the line M_1, with $u_n \neq x$ and $u_n \neq z_1$, such that for the unique path $y_nK_1u_1K_2u_2\cdots K_nu_n$ from the point y_n to the point u_n, holds that $K_1 \neq y_1^\theta$. There exists a path of length $2n+2$ between the flags $\{K_1,u_1\}$ and $\{u_1^\theta,K_1^\theta\}$, namely the path

$$\{K_1,u_1\},\{K_1,y_n\},\{y_1^\theta,y_n\},\{y_1^\theta,L_2^\theta\},\cdots,\{y_n^\theta,L_n^\theta\},\{y_n^\theta,K_1^\theta\},\{u_1^\theta,K_1^\theta\}.$$

Hence the flags $\{K_1,u_1\}$ and $\{u_1^\theta,K_1^\theta\}$ can not lie on a distance smaller than $2n$, otherwise we obtain a k-gon with $k < 2n+1$. Because we assumed that the distance can not be equal to $2n+1$, the flags must lie at distance $2n$. Hence there exists a path $u_1N_{(1,1)}y_{(1,1)}N_{(2,1)}y_{(2,1)}\cdots N_{(n-1,1)}y_{(n-1,1)}u_1^\theta$ between the point u_1 and the line u_1^θ. Note that the point $y_{(n-1,1)}$ is different from K_1^θ (otherwise the points

$$y_{(n-1,1)},\cdots,y_{(1,1)},u_1,y_n,L_2^\theta,\cdots,L_n^\theta,K_1^\theta$$

form a $2n$-gon) and different from the point K_2^θ (otherwise the points

$$y_n,z_n,K_n^\theta,\cdots,K_2^\theta,y_{(n-2,1)},\cdots,y_{(1,1)},u_1$$

form a $2n$-gon). Now we also have a path of length $2n+2$ between the flags $\{K_2,u_2\}$ and $\{u_2^\theta,K_2^\theta\}$ namely the path

$$\{K_2,u_2\},\{K_2,u_1\},\{N_{(1,1)},u_1\},\{N_{(1,1)},y_{(1,1)}\},\cdots,$$
$$\{N_{(n-1,1)},y_{(n-1,1)}\},\{u_1^\theta,y_{(n-1,1)}\},\{u_1^\theta,K_2^\theta\},\{u_2^\theta,K_2^\theta\}.$$

Hence for analogous reasons as above there exists a path

$$u_2N_{(1,2)}y_{(1,2)}N_{(2,2)}y_{(2,2)}\cdots N_{(n-1,2)}y_{(n-1,2)}u_2^\theta$$

between the point u_2 and the line u_2^θ, with $y_{(n-1,2)}$ different from the points K_2^θ and K_3^θ. Going on like this we can finely construct a path

$$u_nN_{(1,n)}y_{(1,n)}N_{(2,n)}y_{(2,n)}\cdots N_{(n-1,n)}y_{(n-1,n)}u_n^\theta$$

between the point u_n and the line u_n^θ, with $y_{(n-1,n)} \neq K_n^\theta$ and $y_{(n-1,n)} \neq z_n$. But now the points $y_{(1,n)},\cdots,y_{(n-1,n)},z_n,\cdots,z_1$ and u_n form a $2n$-gon, a contradiction. $\square$

8
Polar spaces

In this chapter we investigate certain J-domestic collineations of polar spaces. We do not obtain a full classification of all (chamber-)domestic collineations, but we prove some basic results and lay the fundaments that eventually should lead to such a classification, or at least to the classification of fixed point structures of chamber-domestic collineations. In particular we describe in detail the fixed point structures of collineations that are i-domestic and at the same time $(i+1)$-domestic, for all suitable types i. We also show that {point, line}-domestic collineations are either point-domestic or line-domestic, and then we nail down the structure of the fixed elements of point-domestic collineations and of line-domestic collineations. We also show that $\{i, i+1\}$-domestic collineations are either i-domestic or $(i+1)$-domestic (under the assumption that $i+1$ is not the type of the maximal subspaces if i is even). All our results hold in the general case (finite or infinite) with the exception of polar spaces of rank 2, which were treated in detail already in Chapters 6 and 7. For polar spaces of rank at least 3, the ones with lines of size 3 or 5 do not generate special cases or counter examples.

The general philosophy will again turn out to be that domesticity implies a large fixed point structure (or, for point-domesticity, a large fixed element structure). In particular, if a collineation is both i-domestic and $(i+1)$-domestic, then it does not only map no i-space to an opposite one, it always fixes at least one point in every i-space and so they map any i-space to a non-disjoint one. This is again in accordance with our general findings that, if a collineation does not map any flag of certain type to an opposite one, then the maximal distance between a flag of that type and its image is much smaller than opposition.

Also, in classifying point-domestic collineations, Tits-diagrams will turn up. Hence, for the first time, certain fixed point buildings of non-split type relate to domestic automorphisms.

The chapter is structured as follows. First we show that {point,line}-domestic collineations are either point-domestic or line-domestic. Also, a point-domestic collineation which is also line-domestic is necessarily the identity. Then we take a closer look at point-domestic collineations and show that these relate to Tits-diagrams in the non-symplectic case. For line-domestic collineations we show that they always fix a geometric hyperplane pointwise (and vice versa). Finally, we consider i-domestic collineations which are also $(i+1)$-domestic, for $i > 1$, and we also show that, roughly, $\{i, i+1\}$-domestic collineations are either i-domestic or $(i+1)$-domestic.

We repeat that, with a *collineation* of a polar space we mean a collineation of the geometry. Hence, from the point of view of buildings, this includes non-type preserving automorphisms of buildings of type D_{n+1}. Indeed, we also allow for polar spaces whose B_{n+1}-diagram is thin on the last node. One can translate the results of the present chapter to buildings of type D_{n+1} by reading $(n-1)$-domesticity in the polar space as $\{n_+, n_-\}$-domesticity in the oriflamme complex, and n-domesticity in the polar space as n_+- and n_--domesticity in the oriflamme complex (also considering dualities).

For the rest of this section, Γ will denote a polar space of finite rank, furnished with all its projective subspaces. Usually we will assume that the rank of Γ is equal to $n+1$, so that the projective dimension of the maximal subspaces is n. This convention will exceptionally be interrupted in Section 8.2, where rank n gives better formulations of the results in the statements and in the proofs. In any case, the *type* of an element (a subspace) will always be its projective dimension.

8.1 {point,line}-Domestic collineations

The following lemma will turn out to be very useful. We provide two proofs. One proof uses the result of Leeb [32], the other is independent and somewhat longer but introduces a technique that we shall use later.

Lemma 8.1.1 *Suppose that Γ is a polar space of rank $n+1$ and θ is a point-domestic and line-domestic collineation, then θ is the identity.*

Proof. We will first prove that there are no i-dimensional spaces which are mapped to an opposite i-dimensional space, $2 \leq i \leq n$. Suppose, by way of contradiction, that there exists such a space Ω. Take an arbitrary point x in Ω; this point is mapped to the point x^θ in Ω^θ which is not opposite x. Consider the projection $H_x = \mathrm{proj}_{\subseteq\Omega} x^\theta$ of x^θ into Ω; this is an $(i-1)$-dimensional space containing x. The mapping $x \mapsto H_x$ is clearly a duality of Ω (since it is the composition of the collineation $\Omega \to \Omega^\theta : x \mapsto x^\theta$ and the duality $\Omega^\theta \to \Omega : y \mapsto y^\perp \cap \Omega$), and since $x \in H_x$, it is a domestic duality. By Theorem 5.1.1, this duality is a symplectic polarity (in fact, we only use Lemma 5.1.2 here). Hence, if i is even, we obtain a contradiction. If i is odd (with $i > 1$), then there exists a non-isotropic line L in Ω which is mapped to an opposite $(i-2)$-dimensional space of Ω under the symplectic polarity. It follows that L is mapped to an opposite line under θ, again a contradiction.

First proof. We will now prove that every n-dimensional space is fixed. Suppose, by way of contradiction, that there is an n-dimensional space Ω' which is not fixed. Then the intersection of Ω' with Ω'^θ is an i-dimensional space, with $0 \leq i \leq n-1$. Take an $(n-i-1)$-dimensional subspace U of Ω' disjoint from $\Omega' \cap \Omega'^\theta$ and also disjoint from the pre-image (under θ) of $\Omega' \cap \Omega'^\theta$. Because of the previous part of this proof U cannot be mapped to an opposite subspace. Hence there exists a point $u \in U$ which is collinear with every point of U^θ. But u is also collinear with every point of $\Omega' \cap \Omega'^\theta$. Since U^θ is disjoint from $\Omega' \cap \Omega'^\theta$ by the above conditions on U, we see that U^θ and $\Omega' \cap \Omega'^\theta$ generate Ω'^θ, and it follows that u is collinear with all points of Ω'^θ. Hence $u \in \Omega'^\theta$, contradicting $u \in U$ and U disjoint from Ω'^θ.

Second proof. We have proved above that θ is i-domestic, for every type i. This means that no flag whatsoever can be mapped onto an opposite flag, which contradicts the result of Klein & Leeb [32] if θ is not the identity, see also Abramenko & Brown [1].

$\square$

Theorem 8.1.2 *Suppose that Γ is a polar space of rank $n+1 > 2$ and θ is a {point, line}-domestic collineation. Then θ is either point-domestic or line-domestic.*

Proof. Suppose θ is not point-domestic and consider a point x which is mapped to an opposite point x^θ. Take a line L through x and consider the unique line L^φ through x intersecting L^θ in a point. Because θ is {point, line}-domestic, L and L^θ can not be opposite lines. Hence there exists a point y on L^θ which is collinear to all points of L (here collinearity also includes equality). This point should be the intersection of L^θ and L^φ, because it is the only point on L^θ collinear with x. Hence L and L^φ are not opposite in $\mathrm{Res}_\Gamma(x)$, which means that the collineation in the residue of x corresponding with φ is point-domestic.

Consider a plane α of Γ through x. Suppose that α and α^θ are opposite. Take a flag $\{y, M\}$ in α. This flag can not be opposite its image $\{y^\theta, M^\theta\}$. Consider the projection $\{\mathrm{proj}_{\subseteq\alpha} M^\theta, \mathrm{proj}_{\subseteq\alpha} y^\theta\}$ of $\{y^\theta, M^\theta\}$ into α, this is a flag which is not opposite in α to the flag $\{y, M\}$. Hence we obtain a {point, line}-domestic duality in α, a contradiction to Theorem 5.1.1. Hence α and α^θ are not opposite. This means that there exists a point z in α^θ which is collinear to all points of α. Similarly as above, the point z should be in the intersection of α^θ and the unique plane α^φ through x intersecting α^θ in a line. Hence α and α^φ are not opposite in the residue of x, which means that the collineation in the residue of x corresponding with φ is line-domestic. Because of Lemma 8.1.1, this collineation is the identity.

Let z be an arbitrary point in $x^\perp \cap (x^\theta)^\perp$ and let π be an arbitrary plane containing x and z. Also, let x' be a point of π not on the line xz and not collinear with x^θ. By the foregoing, the line xz is mapped under φ to itself, which means that θ maps xz to $x^\theta z$. Our choice of x' implies that x' is opposite x'^θ, and hence z also belongs to $x'^\perp \cap (x'^\theta)^\perp$. Consequently, letting x' play the role of x above, we also have that θ maps $x'z$ to $x'^\theta z$. Hence the intersection z of xz and $x'z$ is mapped to the intersection z of $x^\theta z$ and $x'^\theta z$. We have shown that z is fixed under θ. Hence θ fixes $x \cap x^\theta$ pointwise.

Now consider an arbitrary line K. If K intersects $x^\perp \cap (x^\theta)^\perp$, then it contains at least one fixed point and hence is not mapped onto an opposite line. If K does not intersect $x^\perp \cap (x^\theta)^\perp$, then there exists a line N through x which intersects K in a point y. Since xy is mapped to $x^\theta y'$, with $y' = \mathrm{proj}_{\subseteq xy} x^\theta$, we see that y and y^θ are opposite. Hence we can let y play the role of x above and conclude that K has a fixed point and consequently cannot be mapped onto an opposite line. So θ is line-domestic. $\square$

8.2 Point-domestic collineations

In this section we assume that θ is a point-domestic collineation of the polar space Γ of rank n, with $n > 2$.

Lemma 8.2.1 *The orbit of a point x under the collineation θ is contained in a projective subspace of Γ.*

Proof. We first show by induction on ℓ that the set $\{x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^\ell}\}$ is contained in a subspace. For $\ell = 1$, this is by definition of point-domestic. Now suppose that

$\{x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^{\ell-1}}\}$ is contained in some subspace X, and we may assume that X is generated by $x, x^\theta, x^{\theta^2}, \ldots, x^{\theta^{\ell-1}}$. Applying θ, we see that also $\{x^\theta, x^{\theta^2}, \ldots, x^{\theta^\ell}\}$ is contained in some subspace, namely, X^θ. Consider the line $L := xx^{\theta^{\ell-1}}$, which is mapped onto the line $L^\theta = x^\theta x^{\theta^\ell}$. Consider a point z on L distinct from x and from $x^{\theta^{\ell-1}}$. Then z is collinear to z^θ on L^θ. Since z is also collinear with x^θ (as both points belong to X), we see that z is collinear to x^{θ^ℓ}. Since x^{θ^ℓ} is also collinear to $x^{\theta^{\ell-1}}$, it is collinear with all points of L and hence also with x. This shows that X and X^θ are contained in a common subspace.

This already proves the lemma for θ of finite order. Now suppose the order of θ is infinite. Since the rank of Γ is finite, there exists some natural number k such that x^{θ^k} is contained in the subspace Y generated by $x, x^\theta, \ldots, x^{\theta^{k-1}}$. It is now clear that θ stabilizes Y, as $x^\theta, x^{\theta^2}, \ldots, x^{\theta^k}$ generates a subspace Y^θ contained in Y and of the same dimension as Y (hence coinciding with Y). Consequently the orbit of x generates Y. $\square$

We have now reduced the problem to a geometric one: classify closed configurations of polar spaces whose union is the whole point set. With *closed configuration*, we here understand a set of subspaces closed under projection. Note that this implies closedness of intersection (of intersecting subspaces) and generation (of two subspaces contained in a common subspace).

Usually, such configurations can be rather wild, unless every member has an opposite in the configuration, in which case the configuration forms a building itself. In this case, there is a Tits diagram, and so the types of the elements of the configuration behave rather well. But if some member has no opposite in the configuration, then there is no reason to believe that these types follow certain rules. We shall see examples below. However, the extra condition that every point is contained in some member of the configuration forces the types of elements to obey the same rules as the Tits diagrams, at least when Γ is not a symplectic polar space, i.e. when Γ is not of type C_n.

Theorem 8.2.2 *Let Ω be a set of subspaces of a polar space Γ closed under projection and such that every point is contained in some member of Ω. Assume that Γ is not symplectic. Then there exists a unique natural number i such that the type of each member of Ω is equal to $mi-1$, for some integer m, with i a divisor of n, and m ranging from 1 to n/i (included). Also, for every m with $1 \le m \le n/i$, there exists at least one subspace of type $mi-1$ belonging to Ω, and for every member U of Ω, say of type $ti-1$, and for every m, $1 \le m \le n/i$, there exists a subspace of type $mi-1$ belonging to Ω and incident with U.*

Proof. We prove the assertions by induction on n. Despite the fact that we assume that Γ has rank at least 3, we can include the case $n = 2$ and start the induction with $n = 2$.

For $n = 2$, all assertions follow from the fact that we are dealing with a dual geometric hyperplane as follows from Lemma 7.1.7.

So we may assume $n > 2$. We define i as the smallest positive integer for which there exists a member of Ω of type $i-1$, and we let $U \in \Omega$ be of type $i-1$. If $i = n$, then our assumptions readily imply that Ω consists of a spread and all assertions follow. So we may assume from now on that $i \leq n-1$. Now let Ω_U be the set of all members of Ω containing U. Then clearly Ω_U is closed under projection. We now show that every point of $\mathrm{Res}_\Gamma(U)$ is contained in a member of Ω_U. Hence let W be a subspace of type i containing U. Pick a point x in $W \setminus U$. By assumption, there is some member U' of Ω containing x. Then the subspace $\mathrm{proj}_{\supseteq U} U'$ belongs to Ω and contains both U and x, hence W.

Note that the same argument can be applied to any element of Ω, and in particular, by repeated application, it proves that every element of Ω is contained in a maximal subspace belonging to Ω.

Consequently, for $i \leq n-2$, we can apply induction in $\mathrm{Res}_\Gamma(U)$ and obtain a natural number j such that the type of each member of Ω_U (in Γ) is equal to $i + mj - 1$, for some integer m, with j a divisor of $n - i$, and m ranging from 1 to $(n-i)/j$ (included). Also, for every m with $1 \leq m \leq (n-i)/j$, there exists at least one subspace of type $i + mj - 1$ belonging to Ω_U, and for every member W of Ω_U, say of type $i + tj - 1$, and for every m, $1 \leq m \leq (n-i)/j$, there exists a subspace of type $i + mj - 1$ belonging to Ω_U and incident with W.

If $i = n-1$, then the first assertion in the previous paragraph still holds setting $j = 1$. The second assertion is trivially true. Hence, for now, we do not need to consider the case $i = n-1$ separately.

Consider a point x of Γ not collinear to at least one point of U, and let U_x be a member of Ω containing x (guaranteed to exist by assumption on Ω). By a previous note above, we may assume that U_x has dimension $n-1$. We note that U_x cannot contain U, as x is not collinear to all points of U. Also, U_x is disjoint from U by minimality of i. Hence, as $i \leq n-1$, the subspace $\mathrm{proj}_{\subseteq U_x} U$ is a proper nonempty subspace of U_x disjoint from U and belonging to Ω (nonempty, because the dimension of U is strictly smaller then the dimension of U_x). So the subspace $U' := \mathrm{proj}_{\supseteq U} U_x$ belongs to Ω_U. The induction hypothesis implies that there exists some subspace U'' of (minimal) type $i+j-1$ belonging to Ω_U and incident with (hence contained in) U'. The intersection $V := U'' \cap \mathrm{proj}_{\subseteq U_x} U$ has minimal dimension $j-1$ and belongs to Ω. Minimal here means that every subspace of Ω, all of whose points are collinear with all points of U, has dimension at least $j-1$.

1. First we assume $i + j < n$. Note that the minimal dimension of a subspace of Ω, all of whose points are collinear to all points of U'', is $i - 1$. Indeed, U is such a subspace and the minimality follows from the minimality of i. Consequently, if we interchange the roles of U and V, we also interchange the roles of i and j (the minimality of i is responsible for the fact that the previous paragraphs also hold for j).Hence, looking from both points of view, the subspace of minimal dimension at least $i + j$ (which exists due to $i + j < n$) containing both U and V must have dimension $i + 2j - 1$ and at the same time $j + 2i - 1$. This implies $i = j$ and there only remains to prove the last assertion.

   We first show that any element W of Ω contains a member of Ω of dimension $i - 1$. Indeed, let U be the above member of Ω of type $i - 1$. We may suppose that U is not contained in W and so U and W are disjoint, by minimality of i. We may also assume that the dimension of W is larger than $i - 1$. Then $W' = \mathrm{proj}_{\subseteq W} U$ is nonempty (again because the dimension of U is strictly smaller than the dimension of W) and belongs to Ω. It suffices to show that W' contains an element of type $i - 1$ of Ω. But this now follows from the induction hypothesis by considering a subspace of dimension $2i - 1$ of Ω_U contained in the subspace generated by U and W'.

   Next we show that every subspace of dimension $i - 1$ belonging to Ω is contained in a subspace of dimension $2i - 1$ belonging to Ω. Indeed, let $U' \in \Omega$ be a subspace of dimension $i - 1$, distinct from U (and hence disjoint from it, too). If U and U' are contained in a common subspace, then $\langle U, U' \rangle$ meets the requirement. Otherwise, let $H \in \Omega$ be a maximal subspace through U'. Then the induction hypothesis ensures that there exists a subspace $W \in \Omega_U$ of dimension $2i - 1$ contained in the subspace $\mathrm{proj}_{\supseteq U} H$. The intersection $W \cap H$ has dimension $i - 1$ and so $\langle U', W \cap H \rangle$ meets our requirement (indeed, U' is not contained in W because otherwise U and U' would be contained in a common subspace; U' is disjoint from W by minimality of i).

   It now follows that we can interchange the roles of U with any member V of Ω of type $i - 1$. In particular, Ω_V satisfies the assumptions of our theorem and contains elements of type $mi - 1$, for every $m \in \{2, 3, \ldots, n/i\}$. Hence the last assertion of the theorem follows from first constructing a member $V \in \Omega$ of type $i - 1$ inside a given member W of Ω, and then apply the induction hypothesis to Ω_V to obtain a member of any dimension $mi - 1$, $m \in \{2, 3, \ldots, n/i\}$, incident with both V and W, but in particular W.

2. Next we suppose that $i + j = n$ (and so $U' = U''$ is a maximal subspace). Then

$\mathrm{proj}_{\subseteq U_x} U$ already has dimension $j-1$, and so, by minimality of i, we have $i \leq j$. If $i = j$, there is nothing left to prove. If $i < j$, then consider a point z not collinear to all points of U, and not collinear to all points of V (z can be obtained by using any hyperplane of U'' that does neither contain U nor V). As before, we know that there is a maximal subspace H of Ω containing z. Since by our choice of z, the subspace H can neither contain U nor V, it is disjoint from both U and V. We claim that H is disjoint from U'. Indeed, suppose H meets U' in some subspace S. By minimality of j, S is disjoint from V and has dimension at least $j-1$. But since $j > n-j$, this is a contradiction. Our claim follows. Projecting U and V into H, we obtain two complementary subspaces U_H and V_H in H of dimension $j-1$ and $i-1$, respectively, belonging to Ω. It is clear that these are the only proper subspaces of H that belong to Ω, as otherwise the projection (with the operator $\mathrm{proj}{\subseteq U'}$) into U' produces a contradiction just like in the proof of our last claim above. Now let H' be any maximal subspace of Γ which belongs to Ω. Then H' meets U' and/or H either in one of the proper subspaces of U and H belonging to Ω, or H' coincides with one of H or U', or it is disjoint from both. If H' is disjoint from one of U' or H, then it contains exactly two proper subspaces belonging to Ω, and they have again dimensions $i-1$ and $j-1$. If H' meets U' in U, and if it meets H nontrivially, then it must meet H in U_H (it can clearly not meet H in V_H because $2j > n$). Also, if H' meets U' in V, then it must meet H in V_H (granted it meets H nontrivially) because no point of U_H is collinear to all points of V. So in any case, H' properly contains two members of Ω, of dimensions $i-1$ and $j-1$. Moreover, the above arguments also show that any member of Ω of dimension $i-1$ and any member of Ω of dimension $j-1$ lie together in a joined maximal subspace of Γ.

Now choose a subspace X in U' of dimension $n-3$ and intersecting U in a subspace of dimension $i-2$ (remember that dimension -1 means the empty subspace) and intersecting V in a subspace of dimension $j-2$ (this is never -1). If we now consider the residue $Q := \mathrm{Res}_\Gamma(X)$, which is a generalized quadrangle, then we see that the projection $\mathrm{proj}_{\supseteq X}\Omega$ of Ω onto X is a dual grid in Q, with the extra property that every point of that generalized quadrangle is incident with some line of the dual grid. The latter implies that there are two opposite points x, y in Q with the property that for all points z in Q opposite x the sets $x^\perp \cap y^\perp$ and $x^\perp \cap z^\perp$ have exactly one point in common. Since Q is a Moufang quadrangle; and in particular has the BN-pair property, we see that this property holds for *all* opposite points x and y. Hence, by [40], Q is a symplectic quadrangle and so Γ is a symplectic polar space. This contradicts our assumptions.

The proof of the theorem is complete. □

In the symplectic case there are plenty of counterexamples to the above theorem. An obvious counterexample is the situation at the end of the proof of the previous theorem, when we take $i = 1$, $j = 3$ and hence $n = 4$.

Note that in the symplectic case we can use the structure of the underlying projective space when looking for the fixed structure of a collineation. In particular an inductive process can be used if two non-maximal opposite subspaces of Γ are fixed: the subspace of the surrounding projective space generated by these two subspaces induces a nondegenerate symplectic polar space of lower rank which is also fixed. Nevertheless, this extra tool seems not to be enough to explicitly classify all possibilities, or to at least give a general and uniform description, as in the non-symplectic case.

8.3 Line-domestic collineations

In this section we assume that $n \geq 2$ and we will prove the following theorem:

Theorem 8.3.1 *Suppose that Γ is a polar space of rank $n + 1$ and θ is a nontrivial line-domestic collineation, then θ fixes pointwise a geometric hyperplane.*

For $n = 1$ this is included in Theorem 7.1.1. We will prove this theorem by using the following lemmas.

Lemma 8.3.2 *Suppose that Γ is a polar space of rank $n + 1$ and θ is a line-domestic collineation which is not point-domestic. Suppose that the point x is mapped to an opposite point x^θ. Then $x^\perp \cap (x^\theta)^\perp$ is fixed pointwise.*

Proof. Consider the mapping φ which maps a line L through x to the unique line L^φ through x intersecting L^θ in a point. This mapping is the composition of the restriction to $\mathrm{Res}_\Gamma(x)$ of θ and the projection from $\mathrm{Res}_\Gamma(x^\theta)$ to $\mathrm{Res}_\Gamma(x)$ using the operator $\mathrm{proj}_{\supseteq x}$. So, φ can be conceived as a collineation of the polar space $\mathrm{Res}_\Gamma(x)$ of rank n. If some line L through x were opposite L^φ in $\mathrm{Res}_\Gamma(x)$, then clearly L would be opposite L^θ, contradicting the fact that θ is line-domestic. Hence L and L^φ are not opposite in $\mathrm{Res}_\Gamma(x)$, which means that the collineation in the residue of x corresponding with φ is point-domestic.

Now take a plane α through x and consider again the collineation φ which maps α to the unique plane α^φ through x intersecting α^θ in a line. Suppose that α and α^θ are opposite. Then the duality of α which maps a line L in α to the projection $\mathrm{proj}_{\subseteq\alpha} L^\theta$ is point-domestic. This is a contradiction, since by Lemma 5.1.2 the only such dualities are symplectic polarities and there are no such polarities in a projective plane. Hence the planes α and α^θ are not opposite and hence there exists a point y in α^θ which is collinear to all points of α. This point should be in $\alpha^\varphi \cap \alpha^\theta$. Hence the collineation corresponding to φ in the residue of x is line-domestic.

Analogously to the third paragraph of the proof of Lemma 8.1.2, we can now prove that $x^\perp \cap (x^\theta)^\perp$ is fixed pointwise (but we leave the details to the interested reader).

This completes the proof of Lemma 8.3.2. $\square$

Lemma 8.3.3 *Suppose that Γ is a polar space of rank $n+1$ and θ is a line-domestic collineation. Then every line of Γ contains at least one fixed point.*

Proof. If θ is point-domestic, then the assertion trivially follows from Lemma 8.1.1, so we can assume that θ is not point-domestic. Take a point x which is mapped to an opposite point x^θ. If a line intersects $x^\perp \cap (x^\theta)^\perp$, then because of Lemma 8.3.2, it contains at least one fix point. Hence we consider a line L which does not intersect $x^\perp \cap (x^\theta)^\perp$. There exists a line M through x which intersects L in a point y. Since by Lemma 8.3.2 xy is mapped to $x^\theta y'$, with $y' = \mathrm{proj}_{\subseteq xy} x^\theta$, we see that y and y^θ are opposite. Hence they can play the same role as x and x^θ and so the lines L and L^θ intersect each other in a fixed point.

Lemma 8.3.4 *Suppose that Γ is a polar space of rank $n+1$ and θ is a line-domestic collineation. If an i-dimensional subspace Ω in Γ, with $0 \leq i \leq n$, is fixed by θ, then Ω is fixed pointwise.*

Proof. Suppose first that Ω is n-dimensional and fixed. If all $(n-1)$-spaces in Ω are fixed, then Ω is fixed pointwise. Hence we may assume that there exists an $(n-1)$-dimensional space H in Ω which is not fixed. Take an n-dimensional space Ω' through H different from Ω and consider a line L in Ω', but not in Ω, intersecting H in a point of $H \setminus (H^\theta \cup H^{\theta^{-1}})$. Because θ is line-domestic, there exists a point x on L^θ which is collinear to all points of L (and note that x obviously does not belong to Ω). But this point is also collinear to all points of the $(n-2)$-dimensional space $H \cap H^\theta$. Since L and $H \cap H^\theta$ are skew, they generate Ω', and so $x \in \Omega' \setminus \Omega$. Hence $L^\theta \subseteq \Omega'$ and so L intersects H in a point of $H \cap H^{\theta^{-1}}$, a contradiction.

Secondly, suppose Ω is $(n-1)$-dimensional and fixed. If Ω is contained in a fixed n-dimensional space, we are already done because of the first paragraph of this proof. So we may assume that there does not exists any fixed n-dimensional space containing Ω. Consider an n-dimensional space Σ containing Ω and take a point x in $\Omega \setminus \Sigma$. The point x is mapped to a point opposite x under θ. The $(n-1)$-dimensional space Ω is contained in $x^\perp \cap (x^\theta)^\perp$. Hence, by Lemma 8.3.2, Ω is fixed pointwise.

Now in general take an i-dimensional space Ω, with $i < n-1$, which is fixed by θ. Consider an $(i+2)$-dimensional space through Ω. Take a line in this space skew to Ω to have a line in $\mathrm{Res}_\Gamma(\Omega)$. This line is not opposite its image, hence the corresponding line in $\mathrm{Res}_\Gamma(\Omega)$ cannot be mapped to an opposite line. Hence the collineation in the residue of Ω which corresponds to θ is line-domestic. By Lemma 8.3.3 and the corresponding result for generalized quadrangles (see Theorem 7.1.1), it follows that there exists at least one $(i+1)$-dimensional space containing Ω which is fixed. We can go on like this until we obtain a fixed $(n-1)$-dimensional space Σ. By the foregoing paragraph, Σ, and hence also Ω, is fixed pointwise.

This completes the proof of the lemma. $\square$

Theorem 8.3.1 now follows from the last two lemmas; indeed, Lemma 8.3.4 says that the set of fixed points is a geometric subspace while Lemma 8.3.3 implies that this subspace is a geometric hyperplane.

8.4 Collineations that are i-domestic and $(i+1)$-domestic

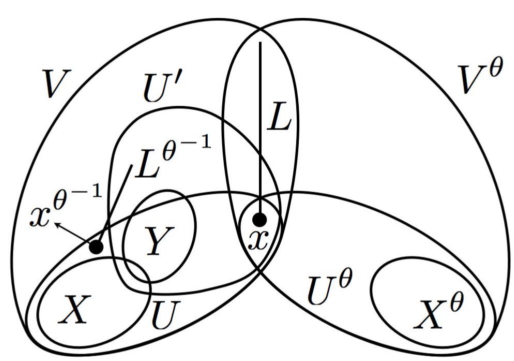
In general, it seems difficult to nail down the fixed point structure of an i-domestic collineation of a polar space, with $i \geq 2$ even. For example, we claim that every point-domestic collineation is i-domestic for all even i. Indeed, if a space U of even positive dimension were mapped onto an opposite, then the duality of U obtained by first applying θ and then $\mathrm{proj}_{\subseteq U}$ is point-domestic. Lemma 5.1.2 implies that this is a symplectic polarity, contradicting the fact that the dimension of U is even. Hence, in this case, in view of Theorem 8.2.2, there should not even be a fixed point! But if i is odd, and θ is an i-domestic collineation, then it is automatically also $(i+1)$-domestic. Indeed, this follows similarly to the second paragraph of the proof of Theorem 8.1.2. Hence, in reality, we classified in the previous section the collineations which are both line-domestic and plane-domestic! In this section, we will generalize this to collineations which are both i-domestic and $(i+1)$-domestic, for $i \geq 2$. It does not matter whether i is odd or even, but we will assume that i is minimal with respect to the property of θ being both i- and

$(i+1)$-domestic. Note that we also already treated this question for $i = 0$. Indeed, this is Lemma 8.1.1.

We have the following theorem, which is somehow the counterpart of Theorem 5.2.3 for polar spaces.

Theorem 8.4.1 *Suppose that Γ is a polar space of rank $n+1$ and suppose that θ is an i-domestic and $(i+1)$-domestic collineation, with $n > i \geq 0$, which is not $(i-1)$-domestic if $i > 0$. Then θ fixes pointwise a geometric subspace of corank i. In particular, every i-dimensional space contains at least one fixpoint.*

Proof. We will prove this by induction on i. For $i = 0, 1$ we already proved this in Lemma 8.1.1 and Theorem 8.3.1. Hence we may assume from now on that $i > 1$. In particular, $n+1 > 3$ and so Γ is an *embeddable* polar space (meaning, it arises from a form in a vector space and so it can be viewed as a substructure of a projective space). Since by assumption θ is not $(i-1)$-domestic, there exists a projective subspace X of type $i-1$ which is opposite its image X^θ. Consider an i-dimensional space U through X and consider the mapping φ which maps the i-dimensional space U to the unique i-dimensional space U^φ through X which is the projection $\mathrm{proj}_{\supseteq X} U^\theta$ of U^θ onto X. Because θ is i-domestic, it follows that U and U^θ are not opposite and one verifies easily that this implies that U and U^φ are not opposite in $\mathrm{Res}_\Gamma(X)$. This means that the collineation—which we also denote by φ—in the residue of X corresponding with φ is point-domestic. Similarly, φ is also line-domestic. By induction, or just by Lemma 8.1.1, it follows that φ is the identity. Hence, with U as above, we know that U and U^θ meet in a point x.



We now claim that x is fixed under θ. Indeed, consider an $(i+1)$-dimensional subspace V through U; then V^θ intersects V in a line L through x. Suppose, by way of contradiction, that x is not fixed. Then x^θ is contained in $U^\theta \setminus U$. It is easy to find an i-dimensional subspace U' in V containing x but neither containing L nor $x^{\theta^{-1}}$. Then U' and its image are clearly disjoint. Let y be the intersection of U' with $L^{\theta^{-1}}$. Choose an $(i-1)$-dimensional subspace Y in U' not through x and not through y. Then Y^θ has no point in common in L (and notice that this is also true for Y). If some point z of Y^θ were collinear to all points of Y, then, since it is also collinear with all points of L it would be collinear with all points of V. Since $z \notin L$, this implies that all points of X are collinear to all points of the plane spanned by L and z, and hence to at least one point of X^θ, contradicting the fact that X and X^θ are opposite. This contradiction shows that Y and Y^θ are opposite. Replacing X by Y in the first paragraph of this proof, we deduce that U' meets its image in a point, a contradiction. This now proves our claim.

Hence we have shown that $X^\perp \cap (X^\theta)^\perp$ is fixed pointwise.

Let M be any maximal subspace of Γ incident with X and consider an arbitrary $(i-1)$-space Z contained in M and not incident with any point of M^θ. The foregoing shows that $M \cap M^\theta$ has dimension $n-i$ in M. Hence Z is complementary in M with respect to that intersection. It follows that no point of Z^θ is collinear with every point of Z, as otherwise that point would be collinear with all points of M, contradicting the fact that it is not contained in M. So we have shown that Z is opposite its image Z^θ. We can now play the same game with Z and obtain that $Z^\perp \cap (Z^\theta)^\perp$ is fixed pointwise. We claim that the set $S := (X^\perp \cap (X^\theta)^\perp) \cup (Z^\perp \cap (Z^\theta)^\perp)$ is connected (meaning that, in the incidence graph, one can walk from any vertex corresponding to a point of this set S to any another such vertex only using subspaces all of whose points belong to S). Indeed, both $X^\perp \cap (X^\theta)^\perp$ and $Z^\perp \cap (Z^\theta)^\perp$ are connected, and their intersection contains $M \cap M^\theta$. The claim follows. This implies the following. We know that the rank of Γ is at least 4, so Γ is embeddable. Let Γ live in the projective space Σ (of possibly infinite dimension). Then θ can be extended to Σ and the subspace of Σ spanned by S is pointwise fixed under this extension of θ.

Let $\mathfrak{X}$ be the set of al $(i-1)$-dimensional subspaces of Γ which can be obtained from X by a finite number of steps, where in each step the next subspace is contained in a common maximal subspace with the previous one, and is mapped onto an opposite subspace under θ. It then follows from the previous paragraph that the projective subspace $\mathfrak{S}$ of Σ generated by $\mathfrak{X}$ is pointwise fixed under θ. Hence the intersection $\boldsymbol{\mathfrak{G}}$ of $\mathfrak{S}$ with Γ is a geometric subspace. Since clearly X is disjoint from $\boldsymbol{\mathfrak{G}}$, the corank of $\boldsymbol{\mathfrak{G}}$ is at most i.

Left to prove is the assertion that every i-space has at least one point in common with $\boldsymbol{\mathfrak{G}}$. Let W be any i-dimensional subspace of Γ. For every $Z \in \mathfrak{X}$, define k_Z to be the dimension of $\mathrm{proj}_{\subseteq W} Z$. Let k be the maximum of all k_Z, with Z running through $\mathfrak{X}$. If

$i-k=0$, the assertion is clear since, if $Z \in \mathfrak{X}$, and W and Z are contained in a common subspace M, then $M \cap M^\theta$ has codimension $i-1$. Now suppose that $i-k>0$ and let $Z \in \mathfrak{X}$ be such that $k_Z = k$. Define $\overline{Z} = \mathrm{proj}_{\supseteq Z} W$. Suppose that W does not meet $Z^* := \overline{Z}^\perp \cap (\overline{Z}^\theta)^\perp$. Note that every $(i-1)$-dimensional subspace of $\overline{Z}$ that does not meet Z^* belongs to $\mathfrak{X}$. Indeed, otherwise some point of $\overline{Z}^\theta \setminus Z^*$ would be collinear with all points of Z, and so all points of Z would be collinear to at least one point of Z^θ (using the fact that Z^* and Z^θ are complementary subspaces of $\overline{Z}^\theta$), contradicting the fact that Z and Z^θ are opposite. Now choose $Z' \in \mathfrak{X}$ such that it is contained in $\overline{Z}$, it contains $\mathrm{proj}_{\subseteq W} Z$ and it is disjoint from Z^* (this is easy). Then $Z' \cap W$ is a k-space, and since the dimension of Z' is smaller than the dimension of W, there are points in W outside Z' collinear to all points of Z'. In other words, $k_{Z'} > k$. This contradicts the maximality of k.

Hence W meets Z^* in at least a point, and the assertion follows. □

This now has the following interesting corollaries.

Corollary 8.4.2 *Suppose that Γ is a polar space of rank $n+1$ with an underlying skewfield which is not commutative. Then a collineation θ is i-domestic for some i, with $0 \le i < n$, if and only if θ pointwise fixes some geometric subspace of corank at most $n-1$.*

Proof. Let θ be i-domestic, with $i < n$. If a k-subspace U, with $k > i$, is mapped onto an opposite k-space, then the composition of the restriction to U of θ with the projection (using the operator $\mathrm{proj}_{\subset U}$) onto U is an i-domestic duality of a k-dimensional space projective space, hence a symplectic polarity by Theorem 5.1.1, contradicting our assumption on the underlying skew field. Hence θ is in particular $(i+1)$-domestic. The assertion now follows from Theorem 8.4.1. □

Corollary 8.4.3 *Suppose that Γ is a polar space of rank $n+1$. Then a collineation θ is i-domestic for some odd i, with $0 \le i < n$, if and only if θ pointwise fixes some geometric subspace of corank at most $n-1$.*

Proof. The proof is totally analogous to the proof of Corollary 8.4.2, noting that no projective space of even dimension $i+1$ admits a symplectic polarity. □

Despite the fact that we are not able to handle the cases of i-domestic collineations for even $i > 0$, we mention the following reduction, which is the analogue of Theorem 8.1.2.

Corollary 8.4.4 *Let θ be an $\{i, i+1\}$-domestic collineation of a polar space Γ of rank $n+1$, with $0 \leq i < n$, with $i < n-1$ if i is even. Then θ is either i-domestic or $(i+1)$-domestic. In particular, if i is odd, then it is $(i+1)$-domestic, and if i is even, then θ is either i-domestic, or $(i+1)$-domestic, but always $(i+2)$-domestic.*

Proof. Suppose first that i is odd. Let U be a subspace of dimension $i+1$ and assume that U is mapped onto an opposite subspace. Then our assumption implies that he composition of the restriction to U of θ with the projection (using $\mathrm{proj}_{\supseteq U}$) onto U is an i-domestic duality, hence a symplectic polarity, contradicting $i+1$ even. So θ is $(i+1)$-domestic.

Suppose now that i is even. Then $i < n-1$ and so we can consider $(i+2)$-dimensional subspaces. If such a subspace U were mapped onto an opposite, then, as in the previous paragraph, we would have a symplectic polarity in U, contradicting $i+2$ is even. Now, if θ is not i-domestic, we can consider an i-space X mapped onto an opposite. Completely similar as in the proof of Theorem 8.4.1, one shows that every $(i+1)$-dimensional subspace contains a fixed point, hence cannot be mapped onto an opposite, and so θ is $(i+1)$-domestic. $\square$

We are still far from a complete understanding of all chamber-domestic collineations, but the above is, in our opinion, a good start.

A
Nederlandstalige samenvatting

Deze thesis bestaat uit twee delen. In het eerste deel onderzoeken we verbanden tussen de parameters van bepaalde meetkunden en het aantal punten dat door een automorfisme (collineatie of dualiteit) in die meetkunde op een zekere afstand wordt afgebeeld. In het tweede deel kijken we naar automorfismen die geen elementen (van bepaald type) op elementen op maximale afstand afbeelden, en gaan we kijken welke invloed dit heeft op de eigenschappen van het automorfisme.

A.1 De verplaatsing onder automorfismen in een aantal eindige meetkunden

We vertrekken hierbij van een stelling van Benson [5] die een verband geeft tussen de parameters s en t van een veralgemeende vierhoek Γ van de orde (s,t), het aantal fixpunten f_0 en het aantal punten f_1 dat op een collineair punt wordt afgebeeld door een collineatie θ van Γ, nl.

$$(1+t)f_0 + f_1 \equiv 1 + st \mod s+t.$$

Een natuurlijke vraag die men zich hier kan stellen is de vraag of er een gelijkaardig verband bestaat voor dualiteiten van veralgemeende vierhoeken van orde s. Een dualiteit kan uiteraard geen punten fixeren, maar het kan wel elementen afbeelden op elementen op afstand 1 of 3. We gaan dus na of we meer kunnen zeggen over het aantal punten dat afgebeeld wordt op een incidente rechte en op een rechte op afstand 3. Meer algemeen kan men zich ook afvragen of er gelijkaardige verbanden bestaan voor collineaties en dualiteiten in willekeurige eindige veralgemeende veelhoeken en ook in andere belangrijke klassen van eindige meetkunden, zoals partiële meetkunden, symmetrische designs, schierveelhoeken en partiële vierhoeken.

A.1.1 Veralgemeende veelhoeken

De volgende resultaten bekomen we voor veralgemeende veelhoeken:

Collineaties in veralgemeende vierhoeken

Zoals we reeds vermeldden, hebben we Benson's stelling [5] voor veralgemeende vierhoeken van de orde (s, t):

Stelling A.1.1 *Als f_0 het aantal punten is dat gefixeerd wordt door een automorfisme θ en als f_1 het aantal punten x is waarvoor $x^\theta \neq x \sim x^\theta$, dan geldt er*

$$(1 + t)f_0 + f_1 \equiv 1 + st \mod s + t.$$

Het besluit van deze stelling kunnen we nu ook als volgt schrijven:

$$(1 + t)f_0 + f_1 = k(s + t) + (1 + s)(1 + t).$$

We kunnen hieruit dan onderstaande gevolgen aantonen:

Gevolg A.1.2 *Veronderstel dat $\mathcal{S}$ een veralgemeende vierhoek van orde (s, t) is en dat θ een automorfisme is van $\mathcal{S}$. Als s en t niet relatief priem zijn, dan bestaat er minstens één fixpunt of minstens één punt dat op een collineair punt wordt afgebeeld.*

Gevolg A.1.3 *Veronderstel dat $\mathcal{S}$ een veralgemeende vierhoek is van de orde s en dat θ een niet-triviaal automorfisme is van $\mathcal{S}$. Als s even is, dan kan θ geen enkele ovoïde en geen enkele dunne deelvierhoek van orde $(1, s)$ puntsgewijze fixeren.*

Collineaties in veralgemeende zeshoeken en dualiteiten in projectieve vlakken

Voor veralgemeende zeshoeken kunnen we een gelijkaardig resultaat bewijzen, waarbij een natuurlijk getal m *compatibel* wordt genoemd met een natuurlijk getal n als $\sqrt{m}$ ofwel niet behoort tot de n-de cyclotomische velduitbreiding van $\mathbb{Q}$, ofwel behoort tot $\mathbb{Q}$.

Stelling A.1.4 *Veronderstel dat $\mathcal{S}$ een veralgemeende zeshoek van de orde (s,t) is en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat st compatibel is met de orde van θ (wat automatisch zo is wanneer $\mathcal{S}$ dik is). Als f_0 het aantal punten is dat gefixeerd wordt door een automorfisme θ, f_1 het aantal punten x waarvoor $x^\theta \neq x \sim x^\theta$ en f_2 het aantal punten dat op een punt op afstand 4 wordt afgebeeld, dan bestaan er gehele getallen k_1 en k_2, waarvoor de volgende gelijkheden gelden:*

$$\begin{aligned} &k_1(s+t+\sqrt{st}) + k_2(s+t-\sqrt{st}) + (1+s)(1+t) = (1+t)f_0 + f_1, \\ &k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + ((1+s)(1+t))^2 \\ &\qquad\qquad\qquad\qquad = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2. \end{aligned}$$

We bekomen hieruit de onderstaande gevolgen:

Gevolg A.1.5 *Veronderstel dat $\mathcal{S}$ een dikke veralgemeende zeshoek van de orde (s,t) is en dat θ een automorfisme is van $\mathcal{S}$. Als s en t niet relatief priem zijn, dan bestaat er minstens één fixpunt of minstens één punt dat op een collineair punt wordt afgebeeld.*

Gevolg A.1.6 *Veronderstel dat $\mathcal{S}$ een dikke veralgemeende zeshoek van de orde (s,t) is en dat θ een involutie is van $\mathcal{S}$. Als s en t niet relatief priem zijn, dan bestaat er minstens één fixpunt of minstens één fixrechte.*

Gevolg A.1.7 *Beschouw een dualiteit θ in een projectief vlak van de orde t. Zij g_1 het aantal absolute punten van θ. Als t geen kwadraat is, maar wel compatibel is met de orde van θ, dan geldt $g_1 = 1+t$. Als t wel een kwadraat is, dan geldt $g_1 \equiv 1 \mod \sqrt{t}$. In het bijzonder is er dus minstens één absoluut punt en bijgevolg ook één absolute rechte.*

Gevolg A.1.8 *Veronderstel dat $\mathcal{S}$ een veralgemeende zeshoek is van de orde s en dat θ een niet-triviaal automorfisme is van $\mathcal{S}$. Als s een veelvoud is van 3, dan kan θ geen enkele ovoïde en geen enkele dunne deelzeshoek van orde $(1,s)$ puntsgewijze fixeren.*

Collineaties in veralgemeende achthoeken en dualiteiten in veralgemeende vierhoeken

Voor veralgemeende achthoeken krijgen we het volgende gelijkaardig resultaat:

Stelling A.1.9 *Veronderstel dat $\mathcal{S}$ een veralgemeende achthoek van de orde (s,t) is en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat $2st$ compatibel is met de orde van θ (wat automatisch zo is wanneer $\mathcal{S}$ dik is). Als f_i, met $i = 0,1,2,3$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$ dan gelden voor zekere gehele getallen k_1, k_2 and k_3 de volgende gelijkheden:*

$$\begin{aligned}
&k_1(s+t+\sqrt{2st}) + k_2(s+t-\sqrt{2st}) + k_3(s+t) + (1+s)(1+t) = (1+t)f_0 + f_1,\\
&k_1(s+t+\sqrt{2st})^2 + k_2(s+t-\sqrt{2st})^2 + k_3(s+t)^2 + ((1+s)(1+t))^2\\
&\qquad\qquad\qquad\qquad = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2,\\
&k_1(s+t+\sqrt{2st})^3 + k_2(s+t-\sqrt{2st})^3 + k_3(s+t)^3 + ((1+s)(1+t))^3\\
&\qquad\qquad\qquad\qquad = (s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0\\
&\qquad\qquad\qquad\qquad + (s(1+t) + (s-1)^2 + st + 3(1+t)(s-1) + 3(1+t)^2)f_1\\
&\qquad\qquad\qquad\qquad + (2(s-1) + 3(1+t))f_2 + f_3.
\end{aligned}$$

We bekomen hieruit de onderstaande gevolgen:

Gevolg A.1.10 *Veronderstel dat $\mathcal{S}$ een dikke veralgemeende achthoek van de orde (s,t) is en dat θ een automorfisme is van $\mathcal{S}$. Als s en t niet relatief priem zijn, dan bestaat er minstens één fixpunt of minstens één punt dat op een collineair punt wordt afgebeeld.*

Gevolg A.1.11 *Veronderstel dat $\mathcal{S}$ een dikke veralgemeende achthoek van orde (s,t) is en dat θ een involutie is van $\mathcal{S}$, dan bestaat er minstens één fixpunt of minstens één fixrechte.*

Gevolg A.1.12 *Veronderstel dat θ een dualiteit is van een veralgemeende vierhoek van de orde t. Als $2t$ geen kwadraat is, maar wel compatibel is met de orde van θ, dan zijn er $1+t$ absolute punten en $1+t$ absolute rechten, en er zijn $(1+t)t^2$ punten die afgebeeld worden op een rechte op afstand 3 en $(1+t)t^2$ rechten die afgebeeld worden op een punt op afstand 3. Als $2t$ een volkomen kwadraat is, dan zijn er $1 \mod \sqrt{2t}$ absolute punten en evenveel absolute rechten.*

Collineaties in veralgemeende twaalfhoeken en dualiteiten in veralgemeende zeshoeken

Hoewel er geen dikke veralgemeende twaalfhoeken bestaan zullen we onderstaande resultaten toch formuleren voor algemene s en t. In werkelijkheid zal echter ofwel $s = 1$ ofwel $t = 1$, maar de formules zullen niet equivalent zijn. Later zullen we dan onze resultaten toepassen op het geval $s = 1$ zodat we resultaten bekomen voor dualiteiten in veralgemeende zeshoeken. In het tweede deel van de thesis zullen we deze resultaten ook toepassen voor $t = 1$.

Stelling A.1.13 *Veronderstel dat $\mathcal{S}$ een veralgemeende twaalfhoek is van de orde (s,t) en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat t en $3st$ compatibel zijn met de orde van θ. Als f_i, met $i = 0,1,2,3,4,5$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$, dan gelden voor zekere gehele getallen k_1, k_2, k_3, k_4 and k_5 de volgende gelijkheden:*

$$
\begin{aligned}
&k_1(s+t+\sqrt{st}) + k_2(s+t-\sqrt{st}) + k_3(s+t+\sqrt{3st}) + k_4(s+t-\sqrt{3st})\\
&\qquad +k_5(s+t) + (1+s)(1+t) = (1+t)f_0 + f_1,\\
&k_1(s+t+\sqrt{st})^2 + k_2(s+t-\sqrt{st})^2 + k_3(s+t+\sqrt{3st})^2 + k_4(s+t-\sqrt{3st})^2\\
&\qquad +k_5(s+t)^2 + ((1+s)(1+t))^2 = (1+s+t)(1+t)f_0 + (1+s+2t)f_1 + f_2,\\
&k_1(s+t+\sqrt{st})^3 + k_2(s+t-\sqrt{st})^3 + k_3(s+t+\sqrt{3st})^3 + k_4(s+t-\sqrt{3st})^3\\
&\qquad +k_5(s+t)^3 + ((1+s)(1+t))^3 = (s(s-1)(1+t) + 3s(1+t)^2 + (1+t)^3)f_0\\
&\qquad +(s(1+t) + (s-1)^2 + st + 3(1+t)(s-1) + 3(1+t)^2)f_1\\
&\qquad +(2(s-1) + 3(1+t))f_2 + f_3,\\
&k_1(s+t+\sqrt{st})^4 + k_2(s+t-\sqrt{st})^4 + k_3(s+t+\sqrt{3st})^4 + k_4(s+t-\sqrt{3st})^4\\
&\qquad +k_5(s+t)^4 + ((1+s)(1+t))^4 = ((s(1+t) + (s-1)^2 + st)(1+t)s\\
&\qquad +4s(s-1)(1+t)^2 + 6s(1+t)^3 + (1+t)^4)f_0 + (s(s-1)(1+t)\\
&\qquad +(s-1)(s(1+t) + (s-1)^2 + st) + 2(s-1)st + 4(1+t)(s(1+t)\\
&\qquad +(s-1)^2 + st) + 6(1+t)^2(s-1) + 4(1+t)^3)f_1\\
&\qquad +(s(1+t) + 3(s-1)^2 + 2st + 8(1+t)(s-1) + 6(1+t)^2)f_2\\
&\qquad +(3(s-1) + 4(1+t))f_3 + f_4,
\end{aligned}
$$

$$
\begin{aligned}
&k_1(s+t+\sqrt{st})^5 + k_2(s+t-\sqrt{st})^5 + k_3(s+t+\sqrt{3st})^5 + k_4(s+t-\sqrt{3st})^5 \\
&\quad +k_5(s+t)^5 + ((1+s)(1+t))^5 = (s(t+1)(s(s-1)(1+t) \\
&\quad +(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st)+5(1+t)(s(1+t)+(s-1)^2 \\
&\quad +st)(1+t)s+10(1+t)^2s(s-1)(1+t)+10(1+t)^3s(t+1)+(1+t)^5)f_0 \\
&\quad +((s(1+t)+(s-1)^2+st)(1+t)s+(s-1)(s(s-1)(1+t) \\
&\quad +(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st)+st(s(1+t)+3(s-1)^2+2st) \\
&\quad +5(1+t)(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st) \\
&\quad +10(1+t)^2(s(1+t)+(s-1)^2+st)+10(1+t)^3(s-1)+5(1+t)^4)f_1 \\
&\quad +(s(s-1)(1+t)+(s-1)(s(1+t)+(s-1)^2+st)+2(s-1)st \\
&\quad +(s-1)(s(1+t)+3(s-1)^2+2st)+3st(s-1)+5(1+t)(s(1+t)+3(s-1)^2 \\
&\quad +2st)+10(1+t)^22(s-1)+10(1+t)^3)f_2 \\
&\quad +(s(1+t)+3(s-1)^2+2st+3(s-1)^2+st+5(t+1)3(s-1)+10(t+1)^2)f_3 \\
&\quad +(4(s-1)+5(t+1))f_4+f_5.
\end{aligned}
$$

We bekomen hieruit de onderstaande gevolgen:

Gevolg A.1.14 *Veronderstel dat θ een dualiteit is van een veralgemeende zeshoek van de orde t en veronderstel dat t en $3t$ compatibel zijn met de orde van θ. Als zowel $3t$ als t geen volkomen kwadraat is, dan heeft θ juist $1+t$ absolute punten en $1+t$ absolute rechten, zijn er t^2+t^3 punten die afgebeeld worden op een rechte op afstand 3 en t^2+t^3 rechten die afgebeeld worden op een punt op afstand 3 en zijn er t^4+t^5 punten die afgebeeld worden op een rechte op afstand 5 en t^4+t^5 rechten die afgebeeld worden op een punt op afstand 5. Als t een volkomen kwadraat is, dan zijn er $1 \mod \sqrt{t}$ absolute punten en evenveel absolute rechten; het aantal punten dat afgebeeld wordt op een rechte op afstand 3 (in de incidentiegraaf) is deelbaar door $\sqrt{t}$. Als $3t$ een volkomen kwadraat is, dan zijn er $1 \mod \sqrt{3t}$ absolute punten en evenveel absolute rechten; het aantal punten dat afgebeeld wordt op een rechte op afstand 3 is deelbaar door $\sqrt{3t}$.*

A.1.2 Symmetrische designs en schierzeshoeken

Collineaties in 2-designs

Stelling A.1.15 *Veronderstel dat $\mathcal{D}$ een $2-(v,t+1,\lambda+1)$-design is en dat θ een automorfisme is van $\mathcal{D}$. Als f_0 het aantal punten is dat gefixeerd wordt door θ en als f_1 het aantal punten x is waarvoor $x^\theta \neq x$, dan geldt er voor een zeker geheel getal k_0 dat*

$$k_0(b-\lambda-1)+b+(v-1)(\lambda+1)=bf_0+(\lambda+1)f_1.$$

Op het eerste zicht zou deze stelling bijkomende informatie kunnen geven over collineaties in 2-designs, maar als we $f_1 = v - f_0$ substitueren, volgt dat $k_0 = f_0 - 1$. We bekomen dus niets nieuws. Als we echter een gelijkaardige formule opstellen voor dualiteiten bekomen we wel nieuwe voorwaarden.

Dualiteiten in symmetrische designs

Het dubbele van een symmetrisch $2-(v,t+1,\lambda+1)$-design is een schierzeshoek van de orde $(1,t;\lambda+1)$.

Stelling A.1.16 *Veronderstel dat $\mathcal{S}$ een schierzeshoek is van orde $(1,t;\lambda+1)$ en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat $t-\lambda$ compatibel is met de orde van θ. Als f_i, met $i=0,1,2$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$, dan gelden voor zekere gehele getallen k_1 en k_2 de volgende gelijkheden:*

$$\begin{aligned}
k_1(1+t+\sqrt{t-\lambda})+k_2(1+t-\sqrt{t-\lambda})+2(1+t) &= (1+t)f_0+f_1,\\
k_1(1+t+\sqrt{t-\lambda})^2+k_2(1+t-\sqrt{t-\lambda})^2+(2(1+t))^2 &= (2+t)(1+t)f_0\\
&\quad +(2+2t)f_1+f_2.
\end{aligned}$$

We bekomen hieruit het onderstaande gevolg.

Gevolg A.1.17 *Veronderstel dat θ een dualiteit is van een symmetrisch $2-(v,t+1,\lambda+1)$-design. Als $t-\lambda$ geen kwadraat is, maar wel compatibel is met de orde van θ, dan heeft θ juist $1+t$ absolute punten en $1+t$ absolute rechten. Als $t-\lambda$ een kwadraat is, dan zijn er $1+t \mod \sqrt{t-\lambda}$, of, equivalent, $1+\lambda \mod \sqrt{t-\lambda}$ absolute punten en evenveel absolute rechten.*

Men kan deze stellingen toepassen op enkele concrete voorbeelden. Hieruit blijkt:

Gevolg A.1.18 *Veronderstel dat q een priemmacht is, die geen volkomen kwadraat is, en dat n even is. Veronderstel dat $T=(t_{ij})_{0\le i\le n, 0\le j\le n}$ een niet-singuliere $(n+1)\times(n+1)$-matrix is en dat σ een automorfisme is van $\mathsf{GF}(q)$, met q een macht van het priemgetal p.*

En veronderstel daarnaast dat p compatibel is met de orde van de semi-lineaire afbeelding bepaald door T en het bijbehorend veldautomorfisme σ. Dan heeft de vergelijking

$$\sum_{i,j=0}^{n} t_{ij} x_i x_j^{\sigma} = 0$$

in de onbekenden $x_0, x_1, \ldots, x_n$ precies q^n oplossingen over $\mathsf{GF}(q)$.

Gevolg A.1.19 *Veronderstel dat θ een collineatie is van ofwel een parabolische kwadriek in de projectieve ruimte $\mathsf{PG}(2n, q)$, ofwel een symplectische polaire ruimte in een projectieve ruimte $\mathsf{PG}(2n-1, q)$, met $n \geq 2$. Dan beeldt θ juist $q^{n-2} + q^{n-3} + \cdots + 1 \mod q^{n-1}$ punten af op een collineair of gelijk punt, en dus worden er $0 \mod q^{n-1}$ punten afgebeeld op een opposite punt.*

Stelling A.1.20 *Elke collineatie van elke eindige polaire ruimte met rang ten minste 3 beeldt minstens één punt af op een niet-opposite punt.*

Gevolg A.1.21 *Veronderstel dat θ een collineatie is van een veralgemeende zeshoek van orde s. Dan beeldt θ juist $s + 1 \mod s^2$ punten af op een niet-opposite punt, en dus worden er $0 \mod s^2$ punten afgebeeld op een opposite punt.*

A.1.3 Partiële meetkunden en schierachthoeken

Collineaties in partiële meetkunden

Stelling A.1.22 *Veronderstel dat $\mathcal{S}$ een partiële meetkunde is van de orde (s, t, α), $1 \leq \alpha \leq \min\{s, t\} + 1$, en dat θ een automorfisme is van $\mathcal{S}$. Als f_0 het aantal punten is dat gefixeerd wordt door θ en als f_1 het aantal punten x is waarvoor $x^\theta \neq x \sim x^\theta$, dan geldt er voor een zeker geheel getal k dat*

$$k(s + t + 1 - \alpha) + (1 + s)(1 + t) = (t + 1)f_0 + f_1.$$

We bekomen hieruit onderstaande gevolgen:

Gevolg A.1.23 *Veronderstel dat $\mathcal{S}$ een partiële meetkunde is van de orde (s, t, α) en dat θ een automorfisme is van $\mathcal{S}$. Als s, t en $\alpha - 1$ een gemeenschappelijke deler hebben, verschillend van 1, dan bestaat er minstens één fixpunt of minstens één punt dat op een collineair punt wordt afgebeeld.*

Gevolg A.1.24 *Veronderstel dat $\mathcal{S}$ een partiële meetkunde is van de orde (s,t,α) en dat θ een involutie is van $\mathcal{S}$. Als s, t en $\alpha-1$ een gemeenschappelijke deler hebben, verschillend van 1, dan bestaat er minstens één fixpunt of minstens één fixrechte.*

Dualiteiten in symmetrische partiële meetkunden

Het dubbele van een symmetrische partiële meetkunde van de orde (t,t,α) is een schierachthoek van de orde $(1,t;\alpha,1)$.

Stelling A.1.25 *Veronderstel dat $\mathcal{S}$ een schierachthoek is van de orde $(1,t;\alpha,1)$ en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat $2t+1-\alpha$ compatibel is met de orde van θ. Als f_i, met $i=0,1,2,3$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$, dan gelden voor de gehele getallen k_1, k_2 en k_3 de volgende gelijkheden:*

$$\begin{aligned}
&k_1(1+t)+k_2(1+t+\sqrt{2t+1-\alpha})+k_3(1+t-\sqrt{2t+1-\alpha})+2(1+t)=(1+t)f_0+f_1,\\
&k_1(1+t)^2+k_2(1+t+\sqrt{2t+1-\alpha})^2+k_3(1+t-\sqrt{2t+1-\alpha})^2+(2(1+t))^2\\
&\qquad\qquad=(2+t)(1+t)f_0+(2+2t)f_1+f_2,\\
&k_1(1+t)^3+k_2(1+t+\sqrt{2t+1-\alpha})^3+k_3(1+t-\sqrt{2t+1-\alpha})^3+((1+s)(1+t))^3\\
&\qquad\qquad=(3(1+t)^2+(1+t)^3)f_0+(1+2t+3(1+t)^2)f_1+3(1+t)f_2+\alpha f_3.
\end{aligned}$$

We bekomen hieruit de onderstaande gevolgen:

Gevolg A.1.26 *Veronderstel dat θ een dualiteit is van een partiële meetkunde van de orde (t,t,α), met $2t+1-\alpha$ geen kwadraat, maar wel compatibel met de orde van θ. Dan heeft θ juist $1+t$ absolute punten en $1+t$ absolute rechten, en dan zijn er $(1+t)t^2/\alpha$ punten die afgebeeld worden op een rechte op afstand 3 en $(1+t)t^2/\alpha$ rechten die afgebeeld worden op een punt op afstand 3.*

Gevolg A.1.27 *Veronderstel dat θ een dualiteit is van een partiële meetkunde van de orde (t,t,α), met $2t+1-\alpha$ een kwadraat. Als α oneven is, dan zijn er $(1+\alpha)/2 \mod \sqrt{2t+1-\alpha}$ absolute punten en evenveel absolute rechten. Als α even is, dan zijn er $(1+\alpha+\sqrt{2t+1-\alpha})/2 \mod \sqrt{2t+1-\alpha}$ absolute punten en evenveel absolute rechten.*

Bijvoorbeeld, elke dualiteit van de sporadische partiële meekunde van Van Lint & Schrijver ($(t,t,\alpha)=(5,5,2)$) heeft steeds $0 \mod 3$ absolute punten (hier, $\sqrt{2t+1-\alpha}=3$). Merk op dat de standaard polariteit precies $6=1+t$ absolute punten heeft.

De partiële meetkunde $\mathsf{pg}(C)$, met C een Thas 1974 maximale boog

Zij C een maximale boog in het projectief vlak $\mathsf{PG}(2,q)$. De partiële meetkunde $\mathsf{pg}(C)$ heeft als puntenverzameling de punten van $\mathsf{PG}(2,q)$ die niet tot de maximale boog C behoren en als rechtenverzameling de secanten van C, met natuurlijke incidentie.

In dit deel bewijzen we de volgende stellingen:

Stelling A.1.28 *Veronderstel dat C een Thas 1974 maximale boog is in $\mathsf{PG}(2,q^2)$, die ontstaat uit een ovoïde $\mathcal{O}$ in $\mathsf{PG}(3,q)$, door de punten van de kegel $x\mathcal{O}$ in $\mathsf{PG}(4,q)\backslash\mathsf{PG}(3,q)$, met $\mathsf{PG}(3,q)\subseteq\mathsf{PG}(4,q)$ en $x\in\mathsf{PG}(4,q)\setminus\mathsf{PG}(3,q)$, te beschouwen. Dan is C isomorf met zijn duale C^*, en er bestaat een dualiteit van $\mathsf{PG}(2,q^2)$ die het punt x omwisselt met de rechte $L_\infty=\mathsf{PG}(3,q)$. In het bijzonder is de partiële meetkunde $\mathsf{pg}(C)$ zelfduaal.*

Stelling A.1.29 *De collineatiegroep van $\mathsf{pg}(C)$ wordt geïnduceerd door de collineatiegroep van $\mathsf{PG}(2,q^2)$.*

Opmerking A.1.30 De voorgaande stelling geldt voor alle maximale bogen C in eindige Desarguesiaanse projectieve vlakken en hun corresponderende partiële meetkunden $\mathsf{pg}(C)$.

Stelling A.1.31 *Er bestaan juist twee isomorfisme klassen van partiële meetkunden $\mathsf{pg}(C)$ in $\mathsf{PG}(2,q^2)$, met $q=2^m$, waarbij C een Thas maximale boog in $\mathsf{PG}(2,q^2)$ is, die correspondeert met een Suzuki-Tits ovoïde (met $m>1$ oneven). Elk van die partiële meetkunden is zelfduaal en elke collineatie en dualiteit van $\mathsf{pg}(C)$ wordt geïnduceerd door een collineatie of dualiteit van het projectief vlak $\mathsf{PG}(2,q^2)$. De grootte van de volledige automorfismegroep is $8m(2^m+\epsilon 2^{\frac{m+1}{2}}+1)(2^m-1)$, met $\epsilon\in\{+1,-1\}$.*

A.1.4 Partiële vierhoeken en schiertienhoeken

Collineaties in partiële vierhoeken

Stelling A.1.32 *Veronderstel dat Γ een partiële vierhoek is van de orde (s,t,μ), $1\leq\mu\leq t+1$, met $(s,t)\neq(1,1)$, en dat θ een automorfisme is van Γ. Als f_i, met $i=0,1,2$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$, dan gelden voor zekere gehele getallen k_1 en k_2 de volgende gelijkheden:*

$$\begin{aligned}&k_1\Big(\tfrac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}\Big)+k_2\Big(\tfrac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}\Big)+(s+1)(t+1)=(t+1)f_0+f_1,\\&k_1\Big(\tfrac{s+2t+1-\mu+\sqrt{(s-\mu+1)^2+4st}}{2}\Big)^2+k_2\Big(\tfrac{s+2t+1-\mu-\sqrt{(s-\mu+1)^2+4st}}{2}\Big)^2+((s+1)(t+1))^2\\&\qquad=(s+t+1)(t+1)f_0+(2(1+t)+(s-1))f_1+\mu f_2.\end{aligned}$$

Opmerking A.1.33 Merk op dat in de voorgaande stelling f_2 kan geschreven worden als $v - f_0 - f_1$, met v het totaal aantal punten, en dit geeft ons een bijkomende relatie.

Dualiteiten in symmetrische partiële vierhoeken

Het dubbele van een symmetrische partiële vierhoek van de orde (t, t, μ) is een schiertienhoek van de orde $(1, t; \mu, 1, 1)$.

Stelling A.1.34 *Veronderstel dat $\mathcal{S}$ een schiertienhoek is van de orde $(1, t; \mu, 1, 1)$ en dat θ een automorfisme is van $\mathcal{S}$. We veronderstellen dat, voor $\epsilon = -1, 1$, het getal $2 + 6t - 2\mu + 2\epsilon\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2}$ compatibel is met de orde van θ. Als f_i, met $i = 0, 1, 2, 3, 4$, het aantal punten is dat afgebeeld wordt op een punt op afstand $2i$, dan gelden voor zekere gehele getallen k_1, k_2, k_3 en k_4 de volgende gelijkheden:*

$$
\begin{aligned}
&k_1((t+1) + \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}) + \\
&\quad k_2((t+1) - \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}) + \\
&\quad k_3((t+1) + \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}) + \\
&\quad k_4((t+1) - \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}) + 2(1+t) = \\
&\quad (t+1)f_0 + f_1,
\end{aligned}
$$

$$
\begin{aligned}
&k_1((t+1) + \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^2 + \\
&\quad k_2((t+1) - \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^2 + \\
&\quad k_3((t+1) + \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^2 + \\
&\quad k_4((t+1) - \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^2 + (2(1+t))^2 = \\
&\quad (t+1)(t+2)f_0 + 2(t+1)f_1 + f_2,
\end{aligned}
$$

$$
\begin{aligned}
&k_1((t+1) + \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^3 + \\
&\quad k_2((t+1) - \tfrac{1}{2}\sqrt{2 + 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^3 + \\
&\quad k_3((t+1) + \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu})^3 + \\
&\quad k_4((t+1) - \tfrac{1}{2}\sqrt{2 - 2\sqrt{5t^2 + 2t - 2t\mu + 1 - 2\mu + \mu^2} + 6t - 2\mu}^3) + (2(1+t))^3 = \\
&\quad (t+1)^2(t+4)f_0 + (3t^2 + 8t + 4)f_1 + 3(t+1)f_2 + f_3,
\end{aligned}
$$

$$
\begin{aligned}
&k_1((t+1)+\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_2((t+1)-\tfrac{1}{2}\sqrt{2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_3((t+1)+\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+\\
&\quad k_4((t+1)-\tfrac{1}{2}\sqrt{2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu})^4+(2(1+t))^4=\\
&\quad (t+1)((t+1)^2(t+7)+2t+1)f_0+4(t+1)((2t+1)+(t+1)^2)f_1+\\
&\quad ((3t+1)+6(t+1)^2)f_2+4(t+1)f_3+\mu f_4.
\end{aligned}
$$

Gevolg A.1.35 *Veronderstel dat θ een dualiteit is van een partiële vierhoek van orde (t,t,μ). Als $2+2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ geen kwadraat is en $2-2\sqrt{5t^2+2t-2t\mu+1-2\mu+\mu^2}+6t-2\mu$ geen kwadraat is, en als beide getallen compatibel zijn met de orde van θ, dan heeft θ juist $1+t$ absolute punten en $1+t$ absolute rechten en er zijn $(1+t)t^2$ punten die afgebeeld worden op een rechte op afstand 3 en $(1+t)t^2$ rechten die afgebeeld worden op een punt op afstand 3. Er zijn dus $\frac{t^3(1-\mu)+t^4}{\mu}$ punten die afgebeeld worden op een rechte op afstand 5 en $\frac{t^3(1-\mu)+t^4}{\mu}$ rechten die afgebeeld worden op een punt op afstand 5.*

A.2 Automorfismen met beperkte verplaatsing

In het tweede deel hebben we ons geïnspireerd op een stelling van Brown & Abramenko [2] die zegt dat elk automorfisme van een irreducibel niet-sferisch gebouw een oneindige verplaatsing heeft. Hun methode geeft ook informatie over het sferische geval, zie ook Leeb [32]. Bijvoorbeeld, in rang 2 gebouwen, beeldt elk automorfisme minstens één kamer af op een kamer met co-afstand één, en als de diameter van de incidentiegraaf even (oneven) is, dan beeldt elke dualiteit (collineatie) minstens één kamer af op een opposite kamer. Voor projectieve vlakken wil dit dus zeggen dat elke collineatie minstens één kamer op een opposite kamer afbeeldt. Hoewel we gemakkelijk kunnen aantonen dat ook dualiteiten in projectieve vlakken steeds een kamer afbeelden op een opposite kamer, bestaan er dualiteiten in projectieve ruimten met oneven dimensie, die geen kamers afbeelden op opposite kamers (zie [2]). Zo'n dualiteit zal steeds een symplectische polariteit blijken te zijn. Een automorfisme dat geen enkele kamer op een opposite kamer afbeeldt, zullen we vanaf nu een *'domestic'* automorfisme noemen. In dit tweede deel hebben we als doel om een aantal domestic automorfismen te classificeren. We zullen ook J-domestic automorfismen bestuderen, waarbij J een deelverzameling is van de verzameling van de types van de meetkunde waarin we werken. We noemen een automorfisme *J-domestic* als het geen enkele vlag van type J op een opposite vlag afbeeldt.

A.2.1 Projectieve ruimten

In deze paragraaf karakteriseren we symplectische polariteiten als de enige dualiteiten van projectieve ruimten die domestic zijn. Dit impliceert een volledige karakterisatie van alle J-domestic dualiteiten van willekeurige projectieve ruimten, voor elke deelverzameling J van de types. We zullen ook J-domestic collineaties in projectieve ruimten karakteriseren en classificeren, voor elke J. Dit omvat dus ook alle domestic collineaties.

We kunnen de volgende resultaten bewijzen:

Domestic dualiteiten

Stelling A.2.1 *Elke domestic dualiteit van een projectieve ruimte is een symplectische polariteit. In het bijzonder zijn er geen domestic dualiteiten in projectieve ruimten van even dimensie.*

Gevolg A.2.2 *Veronderstel dat J een deelverzameling is van de type-verzameling van een n-dimensionale projectieve ruimte, $n \geq 2$. Als J geen even elementen bevat, of als n even is, of als het grondveld (indien het gedefinieerd is) niet abels is, dan bestaat er geen J-domestic dualiteit. In alle andere gevallen zijn symplectische dualiteiten de enige J-domestic dualiteiten.*

J-domestic collineaties

We beschouwen hier collineaties van projectieve ruimten. Neem de projectieve ruimte $\mathsf{PG}(n, \mathbb{K})$, met $\mathbb{K}$ een lichaam. Veronderstel dat J een deelverzameling is van de type-verzameling. We zeggen dat J *symmetrisch* is, als er geldt dat wanneer $i \in J$, dan ook $n - i - 1 \in J$. Het is nu duidelijk dat als J niet symmetrisch is, dat dan elke collineatie J-domestic is, want in dat geval is geen enkele vlag van type J opposite aan een vlag van type J. Dus we mogen nu steeds veronderstellen dat J symmetrisch is.

We kunnen aantonen dat een collineatie van $\mathsf{PG}(n, \mathbb{K})$, J-domestic is als en slechts als ze $\{i, n - i - 1\}$-domestic is, met i het grootste element van J zodanig dat $2i < n$. Hiermee hebben we nu de situatie gereduceerd tot symmetrische type-verzamelingen die uit twee elementen bestaan. Met een vergelijkbare techniek kunnen we dit verder reduceren. Hiertoe geven we eerst de volgende definitie: voor $i \leq n-i-1$ zeggen we dat een collineatie *i-$*$-domestic* is, als θ geen deelruimte van dimensie i op een disjuncte deelruimte afbeeldt.

We kunnen nu aantonen dat het volstaat om alle i-$*$-domestic collineaties te classificeren, voor elke $i \leq n-i-1$, om een classificatie te bekomen van alle J-domestic collineaties van $\mathsf{PG}(n, \mathbb{K})$. We kunnen nu veronderstellen dat een gegeven collineatie i-$*$-domestic is, met $i \leq n-i-1$, maar niet j-$*$-domestic, voor elke $j < i$. In dit geval zeggen we dat de collineatie *scherp i-$*$-domestic* is.

Stelling A.2.3 *Een collineatie θ van $\mathsf{PG}(n, \mathbb{K})$ is scherp i-$*$-domestic, $i \leq n-i-1$, als en slechts als het een deelruimte van dimensie $n-i$ puntsgewijze fixeert, maar geen deelruimte van hogere dimensie.*

Gevolg A.2.4 *Een collineatie θ van een n-dimensionale projectieve ruimte, $n \geq 2$, is domestic als en slechts als θ een deelruimte van dimensie minstens $\frac{n+1}{2}$ puntsgewijze fixeert.*

A.2.2 Enkele kleine veralgemeende veelhoeken

In deze paragraaf classificeren we domestic collineaties van een aantal kleine veralgemeende $2n$-hoeken. De ordes die we hier bekijken lijken misschien vrij willekeurig, maar in de volgende paragraaf zal duidelijk worden waarom we in het geval van de vierhoeken juist deze ordes bekijken. We zullen zien dat dit de enige ordes zijn waarvoor er domestic collineaties voorkomen die noch punt- noch rechte-domestic zijn. Er zal ook blijken dat deze *"uitzonderlijke domestic collineaties"* allemaal orde 4 hebben. In het geval van de zeshoeken hebben we echter geen algemeen resultaat. Uit nieuwsgierigheid bekijken we hier toch een aantal lage ordes. In de gevallen die we hier zullen bekijken zullen de domestic collineaties die noch punt- noch rechte-domestic zijn ook van orde 4 blijken te zijn.

Kleine veralgemeende vierhoeken

Stelling A.2.5 *Veronderstel dat Γ een veralgemeende vierhoek is van de orde (s, t) en dat θ een domestic collineatie is van Γ die noch punt- noch rechte-domestic is. Als $(s, t) \in \{(2, 2), (2, 4), (4, 2), (3, 5), (5, 3)\}$, dan is θ uniek (op toevoeging na) en van orde 4.*

Kleine veralgemeende zeshoeken

Onder andere met behulp van onze stellingen uit Deel A.1 kunnen we hier voor de veralgemeende zeshoeken van ordes $(2, 2)$ en $(2, 8)$, en de gekende zeshoek van orde $(3, 3)$ het volgende aantonen:

Stelling A.2.6 *In de split Cayley zeshoek van de orde* $(2, 2)$ *bestaat er, op toevoeging na, juist één domestic collineatie die noch punt- noch rechte-domestic is. Deze collineatie heeft orde* 4.

Stelling A.2.7 *In de split Cayley zeshoek van de orde* $(3, 3)$ *is elke domestic collineatie ofwel punt- ofwel rechte-domestic.*

Stelling A.2.8 *De trialiteitszeshoek van orde* $(8, 2)$ *heeft een toegevoegde klasse van uitzonderlijke domestic collineaties van orde* 4. *Elk van deze collineaties stabiliseert een deelzeshoek van orde* 2, *waarin de collineatie een uitzonderlijke domestic collineatie induceert.*

A.2.3 Veralgemeende veelhoeken

In deze paragraaf classificeren we alle domestic collineaties van veralgemeende vierhoeken. Al deze collineaties zijn ofwel punt- ofwel rechte-domestic, behalve in de drie uitzonderlijke gevallen die voorkomen in de kleine vierhoeken van de orde $(2, 2)$, $(2, 4)$ en $(3, 5)$ (zie vorige paragraaf). Op dualiteit na vallen ze uiteen in drie klassen: ofwel zijn het centrale collineaties, ofwel fixeren ze een ovoïde puntsgewijs ofwel fixeren ze een grote volle deelvierhoek puntsgewijs. Daarnaast bewijzen we hier ook dat er voor veralgemeende $(2n + 1)$-hoeken geen domestic automorfismen bestaan.

Voor veralgemeende $2n$-hoeken ($n \geq 3$) volgt uit een stelling van Leeb [32] dat er geen domestic dualiteiten bestaan. We kunnen ons hier dus beperken tot domestic collineaties. Het is echter niet eenvoudig om hiervoor een volledige classificatie te bekomen. Behalve voor vierhoeken en de kleine gevallen uit de vorige paragraaf is dit probleem tot hier toe nog onopgelost.

We kunnen hier de volgende resultaten bewijzen:

Domestic collineaties in veralgemeende vierhoeken

Stelling A.2.9 *Als θ een domestic collineatie is van een (niet noodzakelijk eindige) veralgemeende vierhoek Γ van orde (s,t) dan hebben we één van de volgende mogelijkheden.*

(i) *θ is ofwel punt-domestic ofwel rechte-domestic.*

(ii) *$(s,t) \in \{(2,2),(2,4),(4,2),(3,5),(5,3)\}$, θ is noch punt- noch rechte-domestic, uniek (op toevoeging na) en van orde 4.*

Als θ rechte-domestic is, dan hebben we één van de volgende mogelijkheden.

(i) *Er zijn geen fixrechten en de fixpunten van θ vormen een ovoïde.*

(ii) *Er zijn fixrechten, maar geen twee opposite. In dit geval is θ een centrale collineatie.*

(iii) *Er zijn twee opposite fixrechten en de fixstructuur is een volle deelvierhoek Γ' van Γ met de bijkomende voorwaarde dat elke rechte buiten Γ' de deelvierhoek Γ' in een uniek punt snijdt. In het eindige geval is dit equivalent met het feit dat Γ' orde $(s,t/s)$ heeft.*

Als θ punt-domestic is, dan hebben we één van de volgende mogelijkheden.

(i) *Er zijn geen fixpunten en de fixrechten van θ vormen een spread.*

(ii) *Er zijn fixpunten, maar geen twee opposite. In dit geval is θ een axiale collineatie.*

(iii) *Er zijn twee opposite fixpunten en de fixstructuur is een ideale deelvierhoek Γ' van Γ met de bijkomende voorwaarde dat elk punt buiten Γ' incident is met een unieke rechte van Γ'. In het eindige geval is dit equivalent met het feit dat Γ' orde $(s/t,t)$ heeft.*

Domestic collineaties in veralgemeende $2n$-hoeken

Stelling A.2.10 *De fixstructuur van een rechte-domestic collineatie van een veralgemeende $4n$-hoek, $n \geq 1$, en de fixstructuur van een punt-domestic collineatie van een veralgemeende $(4n+2)$-hoek, $n \geq 1$, is een ovoïdale deelruimte.*

Stelling A.2.11 *Een ovoïdale deelruimte van een veralgemeende $2n$-hoek is ofwel een afstands-n ovoïde, ofwel de verzameling van punten en rechten op afstand hoogstens n van een gefixeerd element (dat element is een punt als n even is, en een rechte als n oneven is), ofwel een grote volle deelveelhoek.*

Domestic dualiteiten in veralgemeende $(2n+1)$-hoeken

Stelling A.2.12 *Geen enkele dualiteit van een veralgemeende $(2n+1)$-hoek is domestic.*

A.2.4 Polaire ruimten

In deze paragraaf onderzoeken we een aantal J-domestic collineaties van polaire ruimten. We slagen er niet in om een volledige classificatie te bekomen van alle domestic collineaties, maar we bewijzen wel een aantal basisresultaten die eventueel zouden kunnen bijdragen tot zo'n volledige klassificatie of op zijn minst tot de classificatie van de fixpuntstructuren van domestic collineaties. In het bijzonder onderzoeken we in detail de fixpuntstructuren van collineaties die zowel i-domestic als $(i+1)$-domestic zijn voor elke mogelijke i. Al onze resultaten zijn in het algemeen geldig (zowel eindig als oneindig) met uitzondering van polaire ruimten van rang 2 die we reeds uitvoerig behandeld hebben in de twee voorgaande paragrafen.

We kunnen nu de volgende resultaten bewijzen.

{punt, rechte}-domestic collineaties

Stelling A.2.13 *Veronderstel dat Γ een polaire ruimte is van rang $n+1>2$ en dat θ een {punt, rechte}-domestic collineatie is. Dan is θ ofwel punt-domestic ofwel rechte-domestic.*

Punt-domestic collineaties

Stelling A.2.14 *Veronderstel dat Γ een polaire ruimte is van rang $n>2$ en dat θ een punt-domestic collineatie is van Γ. Dan is de fixstructuur van θ een verzameling van deelruimten gesloten onder projectie met de eigenschap dat elk punt van Γ tot minstens één zo een deelruimte behoort.*

Het probleem herleidt zich dus tot het klasseren van verzamelingen deelruimten met de eigenschappen in bovenstaande stelling aangehaald. Dit geeft aanleiding tot *Tits-diagrammen* (indien de polaire ruimte niet van symplectisch type is), zoals volgende stelling aantoont.

Stelling A.2.15 *Veronderstel dat Ω een verzameling van deelruimten is van een polaire ruimte Γ van rang n, gesloten onder projectie en zodanig dat elk punt bevat is in een* *element van Ω. Neem aan dat Γ niet symplectisch is. Dan bestaat er een uniek natuurlijk getal i zodanig dat het type van elk element van Ω gelijk is aan $mi-1$, voor een zeker natuurlijk getal m, met i een deler van n, en m gaande van 1 tot n/i (inbegrepen). Dan bestaat er ook, voor elke m met $1 \leq m \leq n/i$, minstens één deelruimte van type $mi-1$ die tot Ω behoort, en voor elk element U van Ω, stel van type $ti-1$, en voor elke m, $1 \leq m \leq n/i$, bestaat er een deelruimte van type $mi-1$ die tot Ω behoort en incident is met U.*

Rechte-domestic collineaties

Stelling A.2.16 *Veronderstel dat Γ een polaire ruimte is van rang $n+1$ en dat θ een niet-triviale rechte-domestic collineatie is, dan fixeert θ puntsgewijze een geometrisch hypervlak.*

Collineaties die zowel i-domestic als $(i+1)$-domestic zijn

Stelling A.2.17 *Veronderstel dat Γ een polaire ruimte is van rang $n+1$ en veronderstel dat θ een i-domestic en $(i+1)$-domestic collineatie is, met $n > i \geq 0$, die niet $(i-1)$-domestic is voor $i > 0$. Dan fixeert θ puntsgewijze een geometrisch hypervlak van corang i. In het bijzonder bevat elke i-dimensionale ruimte minstens één fixpunt.*

Dit impliceert de volgende resultaten.

Gevolg A.2.18 *Onderstel dat Γ een polaire ruimte is van rang $n+1$ gedefinieerd over een niet-commutatief lichaam. Dan is een collineatie θ i-domestic voor zekere i, met $0 \leq i < n$, als en slechts als θ een geometrische deelruimte van corang ten hoogste $n-1$ puntsgewijs fixeert.*

Gevolg A.2.19 *Onderstel dat Γ een polaire ruimte is van rang $n+1$. Dan is een collineatie θ i-domestic voor een zeker oneven getal i, met $0 \leq i < n$, als en slechts als θ een geometrische deelruimte van corang ten hoogste $n-1$ puntsgewijs fixeert.*

Gevolg A.2.20 *Zij θ een $\{i, i+1\}$-domestic collineatie van een polaire ruimte Γ van rang $n+1$, met $0 \leq i < n$, en $i < n-1$ als i even is. Dan is θ ofwel i-domestic ofwel $(i+1)$-domestic. Meer bepaald, als i oneven is, dan is het $(i+1)$-domestic, en als i even is, dan is θ ofwel i-domestic, ofwel $(i+1)$-domestic, maar ook altijd $(i+2)$-domestic.*

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